

AD-A086 587

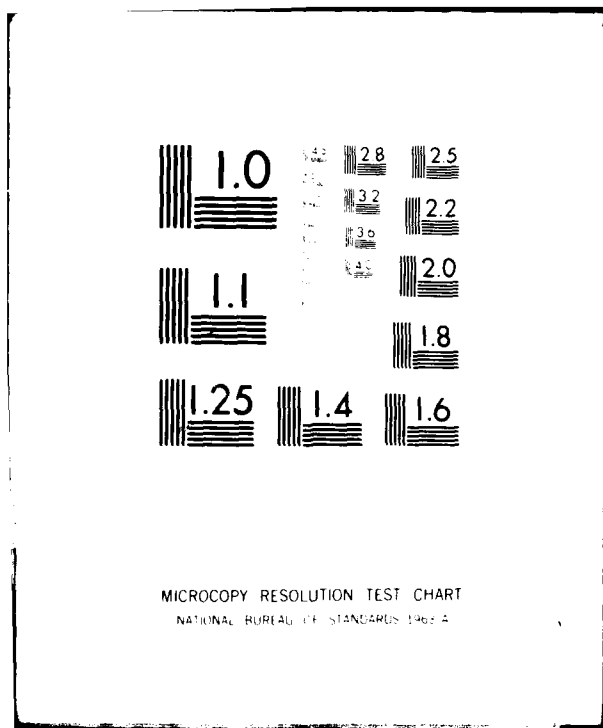
NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CT NEW LO--ETC F/0 20/1
ANALYSIS OF SCATTERING AND RADIATION OF ACOUSTIC WAVES FROM AN --ETC (U)
MAR 80 J S PATEL
NUSC-TR-5821

UNCLASSIFIED

NL

1 OF 1
AD
DISTRIBUTION

END
DATE
FILMED
8-80
DTIC



LEVEL

(12)

NUSC Technical Report 5821

NUSC Technical Report 5821



17 N1

Analysis of Scattering and Radiation of Acoustic Waves From an Axisymmetric Structure In an Infinite Fluid Medium

ADA 086587

Jayant S. Patel
Office of Engineering Mechanics

DTIC
ELECTE
S JUL 10 1980 D
A

27 Mar 1980

12 52

NUSC

Naval Underwater Systems Center
Newport, Rhode Island • New London, Connecticut

Approved for public release; distribution unlimited.

DDC FILE COPY

80 7 9 0 63

PREFACE

This work was accomplished under NUSC Project No. 640P11,
Code 401.

The Technical Reviewer for this report was A. Carlson,
Code 401.

REVIEWED AND APPROVED: 27 March 1980


W. A. VonWinkle
Associate Technical Director
for Technology

The author of this report is located at the New London
Laboratory, Naval Underwater Systems Center,
New London, Connecticut 06320.

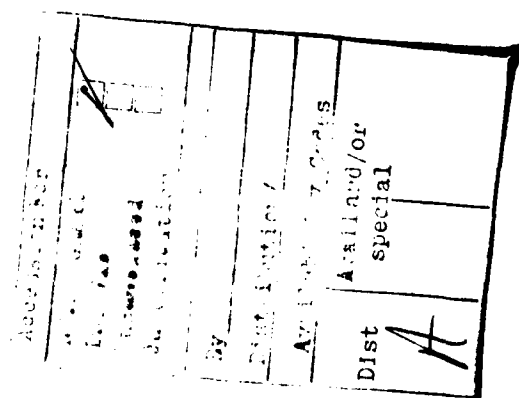
REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR 5821	2. GOVT ACCESSION NO. AD-A086587	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ANALYSIS OF SCATTERING AND RADIATION OF ACOUSTIC WAVES FROM AN AXISYMMETRIC STRUCTURE IN AN INFINITE FLUID MEDIUM		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jayant S. Patel		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Underwater Systems Center New London Laboratory New London, Connecticut 06320		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 640P11
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Underwater Systems Center Newport Laboratory Newport, Rhode Island 02840		12. REPORT DATE 27 March 1980
		13. NUMBER OF PAGES 48
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in BI & 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Helmholtz integrals Cubic polynomial Equations of motion Acoustic radiation Axisymmetric shell Harmonic load Plane wave Plate finite element		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Analysis of scattering and radiation of acoustic waves from an axisymmetric structure in an infinite fluid medium is presented here. It is shown that the use of a basis function, consistent with the one used for axisymmetric pressure on the finite surface, yields accurate results and uses fewer finite surface elements than does piston-type analysis.		

Table of Contents

	Page
Introduction	1
Analysis	2
Solution of Helmholtz Surface Integral	2
Equations of Motion of the Structure	7
Interaction of the Fluid and Structure	9
Application to Computer Programs	13
Summary	16
References	17
Appendix A--Analysis of Axisymmetric Shells	A-1
Appendix B--Analysis of an Incoming Plane Wave at an Angle to the Structure	B-1
Appendix C--Analysis of Scattering of the Plane Wave From Rigid and Flexible Structures	C-1

List of Illustrations

Figure	Page
1 Schematic for Helmholtz Integrals	3
2 Definition of Coordinates of Plate Finite Element	4
3 Pressure Distribution on the Surface of the Sphere for a Given Velocity Distribution $ka = 0.4$	14
A-1 Schematic for Helmholtz Integrals for an Axisymmetric Structure	A-2
A-2 Schematics for Cubic Polynomial	A-6
A-3 Generalized Coordinates	A-17
B-1 Definition of Incident Plane Wave With Respect to Axisymmetric Structures	B-1



Analysis of Scattering and Radiation of Acoustic Waves From an Axisymmetric Structure in an Infinite Fluid Medium

Introduction

Determination of the acoustic radiation from a vibrating surface in an infinite, inviscid fluid medium requires solution both of the equations of motion of the structure and of the Helmholtz equation for the velocity boundary condition normal to the structure's surface. However, the availability of an exact solution of the Helmholtz boundary value is usually precluded because most structures of interest have a complicated surface geometry. As a result, the solution of acoustic radiation problems has generally employed two alternative approaches.

In the first approach¹ the structure is contained within a sphere, and everything within the sphere is regarded as the structure. The surrounding fluid is idealized by fluid finite elements, and the structure is idealized by structural finite elements. The exact solution of the Helmholtz equation for a spherical cavity in an infinite medium is available, and this solution is coupled with the solution for the fluid-structure finite element sphere.

In the second approach² the *integral* equation solution of the Helmholtz boundary value problem is derived. This "surface Helmholtz integral" is derived from the conventional Helmholtz integral for a field point exterior to the cavity by a limiting process in which the field point approaches the surface.

Use of the first approach requires solution of a large fluid-structure finite element problem whenever a long, thin structure is to be analyzed. Use of the second approach yields a nonunique solution in the case where the excitation frequency is within a certain bandwidth around the natural frequency of the interior Helmholtz problem. Two distinct methods^{3,4} are available to alleviate these difficulties. Although they are not detailed here, these methods are incorporated in the computer program.

In earlier works^{2,3,5} on the solution of the Helmholtz surface integral, it is assumed that pressure and velocity over each "finite surface element" are constant; that is, each finite surface element acts as a piston. Our analysis gives the relation between pressure and velocity at the centroid of each finite surface element. The total set of finite surface elements constitutes the total fluid surface area in contact with the structure.

In the earlier works, the response of the structure (normally given at the nodes) and the pressure-velocity relation obtained at the centroids of the finite surface elements are made compatible by interpolating the response at the node points from the response at the centroids. However, since most structures are analyzed using the finite element method, it seems appropriate to divide the fluid surface and the structural surface in contact with each other by the same network of grid lines. In this way, each of the finite surface elements has a counterpart structural finite element on the structural surface. (Previous works do not impose such a restriction.) We can then further require that the velocity and pressure distribution on both sets of elements be consistent. In essence, then, we choose to view the fluid-structure interaction problem as simply an extension of the finite element analysis of structures. This is the new feature examined in this study.

The results obtained thus far show that, for a given criterion of accuracy, the number of finite surface elements required to solve the Helmholtz surface integral is less than that required for structural analysis. This assures us that the finite element model of the structure and its counterpart in fluid is sufficient for determining structural response in an infinite fluid medium.

Although the algebraic development of the work here is restricted to an axisymmetric case, the method can easily be extended to the nonaxisymmetric case.

Analysis

The following analysis is divided into three parts. The first part deals with the numerical solution of the Helmholtz surface integral, the second describes the equations of motion of the structure in the fluid, and the third deals with the response of the structure and scattering of the incident wave train along the axis of symmetry.

Solution of Helmholtz Surface Integral

Radiation from an inclusion in an infinite fluid medium is described by the boundary value problem where the Helmholtz equation represents the field equation:

$$(\nabla^2 + k^2) p(\underline{x}) = 0 \quad (1)$$

subject to the Sommerfeld radiation condition at infinity:

$$\lim_{R \rightarrow \infty} R \left(\frac{\partial p}{\partial R} - ikp \right) = 0 \quad (2)$$

and the pressure $p(\underline{x}')$ and velocity $v_n(\underline{x}')$ normal to the surface prescribed on the inclusion surface.

The solution of the above boundary value problem is given by the Helmholtz integral

$$p(\underline{x}^{(1)}) = \frac{1}{4\pi} \int_A \left\{ p(\underline{x}) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - ik\rho \cdot c \cdot v_n(\underline{x}) \frac{e^{ikR}}{R} \right\} dA, \quad (3)$$

where (see figure 1):

\underline{x}'' , \underline{x} define the vector from the origin of the coordinate system to the field point A and the vector in the fluid to the general surface point B, respectively

\underline{n} is the exterior normal to the surface at point B

$$R = |\underline{x}'' - \underline{x}|$$

$\underline{V}(\underline{x})$ = velocity vector of point B

$V_n(\underline{x}) = \underline{V}(\underline{x}) \cdot \underline{n}$ = component of $\underline{V}(\underline{x})$ normal to the surface

$$k = \omega/c$$

∇^2 = Laplacian operator

ω = frequency of excitation in radians/sec

c = velocity of sound in the fluid

ρ = mass density of the fluid .

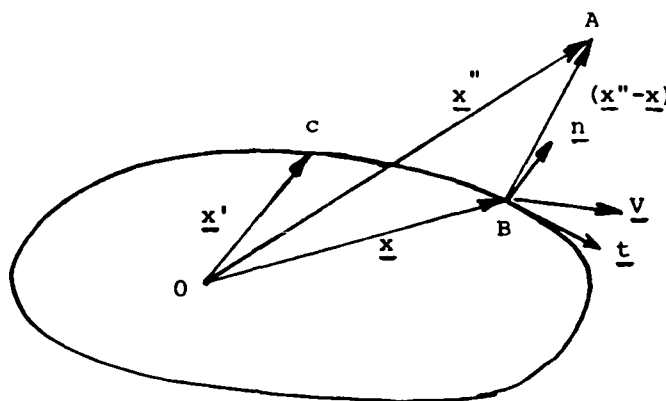


Figure 1. Schematic for Helmholtz Integrals

The Helmholtz integral (equation (3)) gives the pressure at a field point A for the case in which the velocity and pressure distribution on the inclusion surface are given. The problem addressed here is—given a velocity distribution on the inclusion surface, find the corresponding pressure distribution on the surface. This was obtained from equation (3) by considering the limiting case where the field point $A(\underline{x}')$ approaches the surface point $C(\underline{x}')$. This case is considered in detail by Chertock², where the resulting expression, called the Helmholtz surface integral, is an integral equation for $p(\underline{x}')$ when $V_n(\underline{x})$ is prescribed:

$$p(\underline{x}') = \frac{1}{2\pi} \int_A^* \left\{ p(\underline{x}) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - ik\rho c V_n(\underline{x}) \frac{e^{ikR}}{R} \right\} dA \quad (4)$$

where the asterisk (*) over the integral means that the principal value of the improper integral is to be taken on the total surface A. Note that as $R \rightarrow 0$, the integrand $\exp(ikR)/R \rightarrow \infty$, which accounts for the difference between the integrals of equations (3) and (4).

The actual shapes of structures normally encountered rarely conform to the shapes (sphere, cylinder, etc.) for which the exact solution of equation (4) is available. As a result, numerical analysis in which the surface is divided into a set of finite surface elements is used. Assume that the total fluid surface is divided into J number of "finite surface elements"; then the pressure at a grid point $C(\underline{x}')$ is given by

$$p(\underline{x}') = \frac{1}{2\pi} \sum_{j=1}^J \int_{A_j}^* \left\{ p(\underline{x}_j) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - ik\rho c V_n(\underline{x}_j) \frac{e^{ikR}}{R} \right\} dA_j. \quad (5)$$

If the pressure and velocity over each "finite surface element" are regarded as constant (which is the usual assumption), $p(\underline{x}_j)$ and $V_n(\underline{x}_j)$ will drop out of the integral sign, so that

$$p(\underline{x}') = \frac{1}{2\pi} \sum_{j=1}^J \left\{ p(\underline{x}_j) \int_{A_j}^* \frac{\partial}{\partial n} \left(e^{-\frac{ikR}{R}} \right) dA_j - V_n(\underline{x}_j) \int_{A_j}^* ik\rho c \frac{e^{-\frac{ikR}{R}}}{R} dA_j \right\}. \quad (6)$$

Note that \underline{x}' is the centroid of a particular finite surface element and $R = |\underline{x}' - \underline{x}_j|$. Also, it should be clear that both of the integrals in equation (6) will have to be evaluated numerically.

As mentioned earlier, the method under consideration here employs the consistent distribution approach. In the finite element method, the displacement at a point within a particular plate finite element is described by a basis-function (normally a polynomial in local coordinates of the order such that the number of coefficients of the polynomial equal the total number of variables available at the total number of grid points describing the finite element). Consider, for example, the triangle described by three grid points 1, 2, 3 (see figure 2).

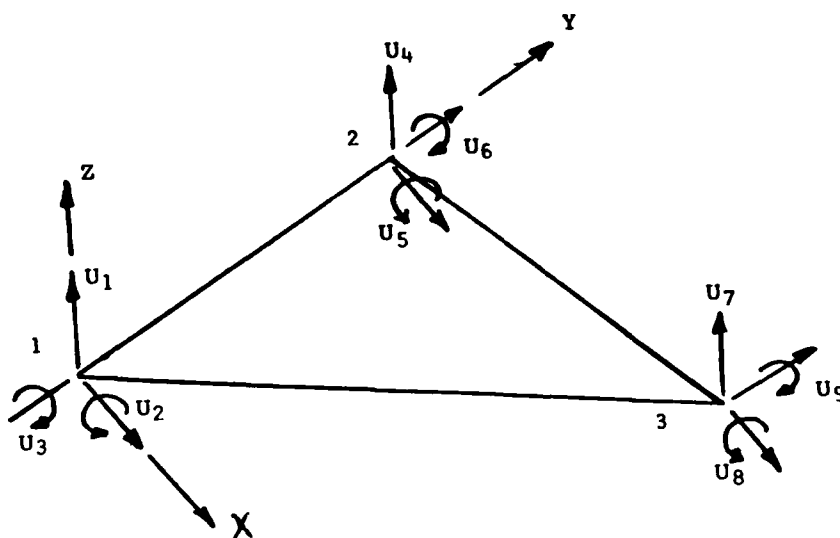


Figure 2. Definition of Coordinates of Plate Finite Element

The terms U_1, U_4, U_7 (the displacements normal to the plane) and the two rotations at each grid point that can produce the normal displacement of the plate account for the total of nine variables. Utilization of these nine variables allows us to select a polynomial in two dimensions (x, y —which is the local coordinate system of the triangular plate) with nine coefficients:

$$u_z = C_1 + C_2x + C_3y + C_4x^2 + C_5xy + C_6y^2 + C_7x^3 + C_8(x^2y + xy^2) + C_9y^3.$$

These nine coefficients can be expressed in terms of nine known quantities. Without deriving the expressions explicitly for the coefficient C_i , we will write u_z in matrix notation:

$$u_z = \sum_{i=1}^9 a_i(x, y) u_i = a^T u, \quad (7)$$

where $a_i(x, y)$ are the polynomials in local coordinates x, y , and

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_9 \end{pmatrix}.$$

Selection of the basis function for an axisymmetric conical shell element is detailed in appendix A.

Note that the nonlinear variation of the displacement in the previous example requires not only the displacements at the grid points, but also the rate of change of the displacements. Since the general purpose computer programs based on the finite element method always give the translational and angular velocities at the grid points, the use of nonlinear distribution does not require any change to the existing computer programs. This is not the case with the pressure distribution problem. Equation (5) gives pressure information only, whereas for the nonlinear pressure distribution case, we also need the rate of change of pressure at the grid points. To obtain this information, we will differentiate equation (5) with respect to the local spatial variables:

$$\frac{\partial}{\partial t_k} p(x') = \frac{1}{2\pi} \sum_{j=1}^J \int_{\Delta j}^* \left\{ p(x_j) \frac{\partial^2}{\partial n \partial t_k} \left(\frac{e^{ikR}}{R} \right) - ik\rho c v_n(x_j) \frac{\partial}{\partial t_k} \left(\frac{e^{ikR}}{R} \right) \right\} d\Delta j, \quad (8)$$

where

$$k = 1, 2$$

n_1, t_1, t_2 form a local right-hand coordinate system at point $C(x')$

n is the normal to the surface at $C(x')$

t_1, t_2 are tangent to the surface at $C(x')$ and are at right angles to each other. Either t_1 or t_2 (whichever is convenient) is fixed to define the local coordinate system (see x, y, z coordinate system in figure 2); the remaining axis should form the right-hand system.

Then, with both $p(x')$ and $\partial/(\partial t_k) p(x')$ known at the grid points, an interpolation function (corresponding to equation (7)) can be written for the pressure.

In accordance with the notations used in equation (7) for the consistent distribution, let us define pressure and velocity for the j th "finite surface element":

$$p(x_j) = a_j^T p_j^{\ell}$$

$$v_n(x_j) = a_j^T v_j^{\ell}, \quad (9)$$

where l refers to the variable defined in local coordinates.

In the analysis we also considered the second case in order to see how much difference (if any) the linear distribution of pressure over the "finite surface elements" would make. In the linear case, the first set of equations of equation (9) will change:

$$\begin{aligned} p(\underline{x}_j) &= c_j^T p_j^\ell \\ v_n(\underline{x}_j) &= a_j^T v_j^\ell \end{aligned} \quad (10)$$

To increase notational flexibility, we will use equation (10) for both cases, with the understanding that for the consistent distribution case $c_j^T = \underline{a}_j^T$ and that for the linear pressure distribution cases, pressure vector \underline{p}_j^ℓ will have half the number of elements in the vectors in equation (9) required for the consistent distribution case and $c_j^T \neq \underline{a}_j^T$.

Substituting equation (10) into equations (5) and (8), we get

$$p(\underline{x}^1) = \frac{1}{2\pi} \sum_{j=1}^J \left\{ \left[\int_{A_j}^* c_j^T \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) dA_j \right] p_j^\ell + \left[-ik\rho c \int_{A_j}^* a_j^T \cdot \frac{e^{ikR}}{R} dA_j \right] v_j^\ell \right\} \quad (11)$$

$$\frac{\partial}{\partial t_k} p(\underline{x}^1) = \frac{1}{2\pi} \sum_{j=1}^J \left\{ \left[\int_{A_j}^* c_j^T \frac{\partial^2}{\partial n \partial t_k} \left(\frac{e^{ikR}}{R} \right) dA_j \right] p_j^\ell + \left[-ik\rho c \int_{A_j}^* a_j^T \frac{\partial}{\partial t_k} \left(\frac{e^{ikR}}{R} \right) dA_j \right] v_j^\ell \right\} \quad (12)$$

Appendix A provides the details of numerical integration to be performed in evaluating the above integrals. Collecting the terms in proper order, we can write equations (11) and (12) in matrix notation:

$$\{P_n\} = [\bar{H}]\{P_n\} + [G]\{V_n\} \quad (13)$$

In the above equations, n indicates that we are dealing only with the components of pressure and velocity normal to the surface.

Let $[E]$ be an identity matrix of the size of $[\bar{H}]$; then we can rewrite equation (13) in convenient form:

$$\begin{aligned} [E]\{P_n\} &= [\bar{H}]\{P_n\} + [G]\{V_n\} \\ [E] - [\bar{H}]\{P_n\} &= [G]\{V_n\} \\ [H]\{P_n\} &= [G]\{V_n\}, \end{aligned} \quad (14)$$

where

$$[H] = [E] - [\bar{H}].$$

Equations of Motion of the Structure

The equations of motion of a structure idealized into a finite element model are written in matrix notation as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F_e\} + \{F_f\}, \quad (15)$$

where

$[M]$ is a mass matrix

$[K]$ is a stiffness matrix

$\{x\}$ is the vector of generalized displacement

$\{F_e\}$ is the vector of generalized mechanical forces

$\{F_f\}$ is the vector of generalized fluid pressure forces normal to the surface .

In practice, stiffness and mass matrices of a structure are assembled by (1) writing element stiffness and mass matrices for each finite element in their local coordinate system, (2) transforming all element matrices individually in the global coordinate system, and (3) arranging and assembling all the matrices in final form.

A generalized set of displacements $\{u\}$ of a material point of a j th finite element is described by a matrix of basis functions:

$$\{u\} = [a] \{U^{\ell}\} ,$$

where

$\{u\}$ is the $(r \times 1)$ vector of the generalized displacements, $r \leq 6$

$[a]$ is the $(r \times k)$ rectangular matrix of basis functions

$\{U^{\ell}\}$ is the vector of $(k \times 1)$ elements, where k = the number of generalized coordinates defining the finite element.

Using the energy principle, we can derive the element mass matrix:

$$[M_j] = \int_{V_j} \rho [a]^T [a] dV_j ,$$

where

V_j is the volume of the finite element

ρ is the mass density of the finite element .

The mass matrix, as defined above, is the consistent mass matrix; however, a lumped mass model can also be used. The difference between the two mass matrices is significant. In the consistent mass

matrix, the deformed configuration used is the same as the one used to derive the stiffness matrix. The lumped mass approach uses a discontinuous deformed configuration in which several parts of a particular finite element tend to vibrate as pistons, each with different amplitudes. The advantage of the consistent mass matrix is that it requires the structure to be divided into fewer finite elements than does the lumped mass approach. The disadvantage is that the consistent mass matrix is nondiagonal, and most of the eigenvalue routines available accept only the diagonal mass matrix. However, since the problem we are concerned with uses the inversion routine, our use of the consistent mass matrix is not a disadvantage. From the previous discussions, it can be seen that the proposed plan to use consistent velocity and pressure distribution to solve the Helmholtz surface integral is a simple extension of the idea which led to the development of the consistent mass matrix.

The generalized forces can be obtained using the virtual work principle. Equating the total virtual work done by the mechanical forces and the generalized forces, we get for the j th finite element:

$$\{F_{fn}^j\}^T \{\delta V_j^L\} = \int_{s_j} p(\underline{x}_j) \delta V_n(\underline{x}_j) ds_j ,$$

where s_j is the surface of the finite element.

Substituting the expression for p and V_n from equation (10), we get

$$\int_{s_j} p(\underline{x}_j) \delta V_n(\underline{x}_j) ds_j = \left\{ \int_{s_j} \{P_j^L\}^T [C_j] [a_j]^T ds_j \right\} \{\delta V_j^L\} ,$$

which results in

$$\{F_{fn}^j\} = \left[\int_{s_j} [a_j] [c_j]^T ds_j \right] \{P_j\} = [A_j] \{P_j\} .$$

The matrix equation of the generalized forces for the entire structure can be written by properly arranging the above element generalized forces:

$$\{F_{fn}\} = [A] \{P_n\} . \quad (16)$$

This expression gives the normal generalized forces in terms of the normal pressure vector.

As explained later in the Solution of Coupled Equations section, we can write

$$\begin{aligned} \{P_n\} &= [T] [B] \{P\} \\ \{F_{fn}\} &= [T] [B] \{F_f\} . \end{aligned} \quad (17)$$

Substituting equation (17) into equation (16), we get

$$\{F_f\} = [B]^T [T]^T [A] [T] [B] \{P\} . \quad (18)$$

Substituting equation (18) into equation (15), we get

$$[M] \{\ddot{x}\} + [K] \{\dot{x}\} = \{F_e\} + [B]^T [T]^T [A] [T] [B] \{P\} . \quad (19)$$

Interaction of the Fluid and Structure

In discussing the interaction of the fluid and structure, we will first consider a simple scattering problem.

Scattering

Before we deal with the solution of equations (14) and (19), let us consider the case where a train of harmonic plane waves traveling along the axis of symmetry impinges on the structure. Superscripts i and s refer to the incident and scattered waves. p^i, \underline{v}^i are the incident pressure and velocity vectors. p^s, \underline{v}^s are the scattered pressure and velocity vectors.

$$\text{Let } \underline{v}^i = v_o e^{j(kx + \omega t)} \underline{i}_x = v^i \underline{i}_x$$

$$\underline{v}^s = v^s \underline{n} \quad (20)$$

$$p = p^i + p^s \quad (21)$$

$$\underline{v}^n = (\underline{v}^i \cdot \underline{n}) \underline{n} + \underline{v}^s. \quad (22)$$

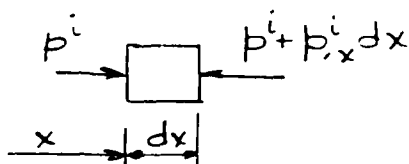
p, \underline{v}^n are the total pressure and velocity vectors normal to the surface of the structure. $\underline{i}_x, \underline{n}$ are the unit vectors, respectively, along the axis of symmetry and normal to the surface.

Scattered components of pressure and velocity, p^s and \underline{v}^s , satisfy the Helmholtz surface integral, yielding

$$[H]\{P^s\} = [G]\{\underline{v}^s\}. \quad (23)$$

Incident pressure and velocity are related by the equation of motion

$$\rho_f \frac{\partial v^i}{\partial t} = \frac{\partial p^i}{\partial x}. \quad (24)$$



For harmonic velocity distribution, it can be shown that

$$p^i(x, t) = -c\rho_f v^i(x, t). \quad (25)$$

Equation (22) relates only the normal components of velocity.

$$\text{Let } \underline{v}^i \cdot \underline{n} = v^{in}$$

$$\underline{v}^n = v^n \underline{n},$$

where v^n is the component of velocity of the structure normal to the surface of the structure. Now we can rewrite equation (22) as

$$v^n = v^{in} + v^s. \quad (26)$$

Substituting equation (26) into equation (23), we get

$$\{p^s\} = [H]^{-1}[G]\{v\}^n - \{v^{in}\}. \quad (27)$$

And substituting equations (25) and (27) into equation (21), we get

$$\{p\} = -c\rho_f\{v^i\} + [H]^{-1}[G]\{v\}^n - [H]^{-1}[G]\{v^{in}\}. \quad (28)$$

The analysis of an incoming plane wave at an angle to the structure is given in appendix B. Appendix C discusses further the scattering of the plane wave from rigid and flexible structures.

Solution of Coupled Equations

We are dealing with acoustic pressure which, by definition, is a small variation in some uniform pressure field. This precludes any possibility of cavitation in the fluid near the structure, and we can assume that the velocity of the structural surface is the same as the velocity of the fluid at that point, and that the pressure exerted on the structural surface is the same as the pressure in the fluid at that point. It is this condition of continuity of pressure and velocity that couples equations (14), (19), and (28).

Before we couple these equations, the pressures and velocity vectors of equations (14) and (28)—which consider only the normal component of velocities of the grid points located on the surface—will have to be transformed to the generalized force and velocity vectors of the entire structure in the basic coordinate system. The set of grid points on the surface of the structure is the subset of all the grid points on the structure. Also, the pressures and velocities in equations (14) and (23) that are normal to the surface of the structure will have to be transformed to the basic coordinate system.

Let $[T]$ be the transformation matrix relating the normal variables to the variables in the basic coordinates. Since it is obtained by rotating each of the local variables, the resulting matrix $[T]$ is orthogonal. After the variables of this subset have been transformed to the basic coordinate system, the subset must be related to the total set. Conceptually, this is performed by a rectangular matrix $[B]$; for example, to relate the vector $\begin{Bmatrix} x_1 \\ x_3 \end{Bmatrix}$ to vector $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$,

we can devise the matrix shown below.

$$\begin{Bmatrix} x_1 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}.$$

Although we will retain the matrix $[B]$ (defined above) in this work, in practice the matrix that $[B]$ is to be expanded to (by inserting proper rows and columns of zeros) is used.

From the above, then, we can write

$$\{v_n\} = [T]\{v^s\} = [T][B]\{v\}. \quad (29)$$

Similarly,

$$\{P_n\} = [T][B]\{P\}. \quad (30)$$

Substituting equations (29) and (30) into equation (14), we get

$$[T][B]\{P\} = [H]^{-1}[G][T][B]\{v\}.$$

Since $[T]$ is orthogonal, i.e., $[T]^{-1} = [T]^T$, the above equation reduces to

$$[B]^T[B]\{P\} = [B]^T[T]^T[H]^{-1}[G][T][B]\{v\}.$$

$[B]^T[B]$ is a diagonal matrix with unity along the diagonal, except for the points (corresponding to the interior points of the structure) on which there is no fluid pressure and which are zeros. In fact, if the above equations are written out explicitly, both sides will be identically zero for the interior points. In what follows we will replace $[B]^T[B]\{P\}$ by $\{P\}$, with the implicit understanding that the new vector includes the pressures at the surface points and the interior points, and that the pressure at the interior points is zero.

We now write

$$\{P\} = [B]^T[T]^T[H]^{-1}[G][T][B]\{v\}. \quad (31)$$

In a similar way we can transform equation (28):

$$\begin{aligned} \{P\} = & -c\rho_f[B]^T[B]\{\underline{v}^i\} + [B]^T[T]^T[H]^{-1}[G][T][B]\{\dot{v}\} - \\ & [B]^T[T]^T[H]^{-1}[G][T][B]\{\underline{v}^i\}. \end{aligned} \quad (32)$$

It is understood here that vector $\{v^i\}$ is defined in the basic coordinate system of the structure for only the surface grid points and that $\{\underline{v}^i\}$ is the vector's counterpart for the total set of grid points. Since it can easily be verified that equation (31) is the subset of equation (32), we will consider the general equation (32).

Substituting equation (32) into equation (19), we get

$$\begin{aligned} [M]\{\ddot{x}\} + [K]\{\dot{x}\} = & \{F_e\} - c\rho_f[T_1]^T[A][T_1]\{\underline{v}^i\} + \\ & [T_1]^T[A][H]^{-1}[G][T_1]\{\dot{x}\} - [T_1]^T[A][H]^{-1}[G][T_1]\{\underline{v}^i\}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} [T_1] &= [T][B] \\ \{\dot{x}\} &= \{\dot{v}\}. \end{aligned}$$

Direct Analysis. Since we are considering only harmonic forces,

$$\begin{aligned}\{F_e\} &= \{F_e^*\} e^{i\omega t} \\ \{x\} &= \{X\} e^{i\omega t} \\ \{\underline{v}_1\} &= \{\underline{v}_1^*\} e^{i\omega t}.\end{aligned}\tag{34}$$

Substituting equation (34) into equation (33), we get

$$\begin{aligned}& \left[-\omega^2 [M] + [K] - i\omega [T_1]^T [A] [H]^{-1} [G] [T_1] \right] \{X\} = \\ & \{F_e^*\} - c\rho_f [T_1]^T [A] [T_1] \{\underline{v}_1^*\} - [T_1]^T [A] [H]^{-1} [G] [T_1] \{\underline{v}_1^*\}.\end{aligned}\tag{35}$$

The solution of equation (35) yields the displacement vector $\{X\}$, which can be substituted back into equation (31) or (32) to get the pressure on the surface.

Modal Analysis. The matrix to be inverted in equation (35) is complex, full, and sometimes ill-conditioned because the terms of the stiffness matrix are large for certain ranges of driving frequencies. Also the imaginary part of the matrix is small compared with the real part. An alternative in this situation is to employ modal transformation.

Let $[\phi]$ be the matrix of the desired N modes, and let these modes be normalized so that

$$[\phi]^T [M] [\phi] = [E], \tag{36}$$

where $[E]$ is the identity matrix. We know then that

$$[\phi]^T [K] [\phi] = \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \\ & & & \lambda_N^2 \end{bmatrix} = [\lambda]^D. \tag{37}$$

Let us use the transformation

$$\{x\} = [\phi] \{q\}, \tag{38}$$

where $\{q\}$ is the vector of the generalized coordinates.

Substituting transformation (38) into equation (33) and premultiplying the resulting equations by $[\phi]^T$, we get, using the relations (36) and (37),

$$\begin{aligned}\{\ddot{q}\} + [\lambda]^D \{q\} &= [\phi]^T \{F_e\} - c\rho_f [\phi]^T [T_1]^T [A] [T_1] \{\underline{v}_1\} + \\ & [\phi]^T [T_1]^T [A] [H]^{-1} [G] [T_1] [\phi] \{\dot{q}\} - \\ & [\phi]^T [T_1]^T [A] [H]^{-1} [G] [T_1] \{\underline{v}_1^1\}.\end{aligned}$$

Since the forces and resulting displacements are all harmonic, the above differential equations reduce to

$$\begin{aligned} & \left[[\lambda]^D - \omega^2 [E] - i\omega [\phi]^T [T_1]^T [A] [H]^{-1} [G] [T_1] [\phi] \right] \{q^*\} = \\ & [\phi]^T \{F_e^*\} - c\rho_f [\phi]^T [T_1]^T [A] [T_1] \{\underline{v}_1^*\} - \\ & [\phi]^T [T_1]^T [A] [H]^{-1} [G] [T_1] \{\underline{v}_1^*\}, \end{aligned} \quad (39)$$

where

$$\begin{aligned} \{q\} &= \{q^*\} e^{i\omega t} \\ \{F_e\} &= \{F_e^*\} e^{i\omega t} \\ \{\underline{v}_1\} &= \{\underline{v}_1^*\} e^{i\omega t}. \end{aligned}$$

The solution of equation (39) will be simpler than that of equation (35) because the matrix is smaller. Using equation (38) to transform the solution $\{q^*\}$ back to $\{X\}$, we get the displacements of the structure, and from the displacements we get the pressures on the surface of the structure.

Examples. Of the many examples solved, the solutions of two problems are presented here. In the first problem, as shown in figure 3, the pressure distribution on the surface of the rigid sphere is calculated for a prescribed velocity distribution. In the second problem, a steel sphere submerged in fluid and excited by a point load is considered.

The series solution⁷ of the acoustic pressure on the surface of a point-driven sphere has been programmed. Fifty terms of the series were retained for the computation. The results are plotted as a series solution in figure 4. Two different finite element models of the sphere were considered. The model used in conjunction with equation (35) represents the sphere by 8 conical shell elements, whereas the model used with equation (34) represents the sphere by 18 conical shell elements.

The first three modes of the 18 element model were retained for the calculations. The intention was to see how well the series solution compares with the Fluid Interacting with Structures (FIST) program, in which relatively crude finite element models are used. The agreement among the three sets is excellent everywhere except in the vicinity of the point load. This is to be expected since even for a dynamic response of the sphere in the vicinity of the load *in vacuo*, a more refined finite element model than the one used here is required.

Application to Computer Programs

The computer program was structured to solve four cases: (1) radiation from a vibrating cavity, (2) scattering of a plane wave traveling along the axis of symmetry from a rigid cavity, (3) radiation from a vibrating structure, and (4) scattering of a plane wave traveling along the axis of symmetry from a flexible structure.

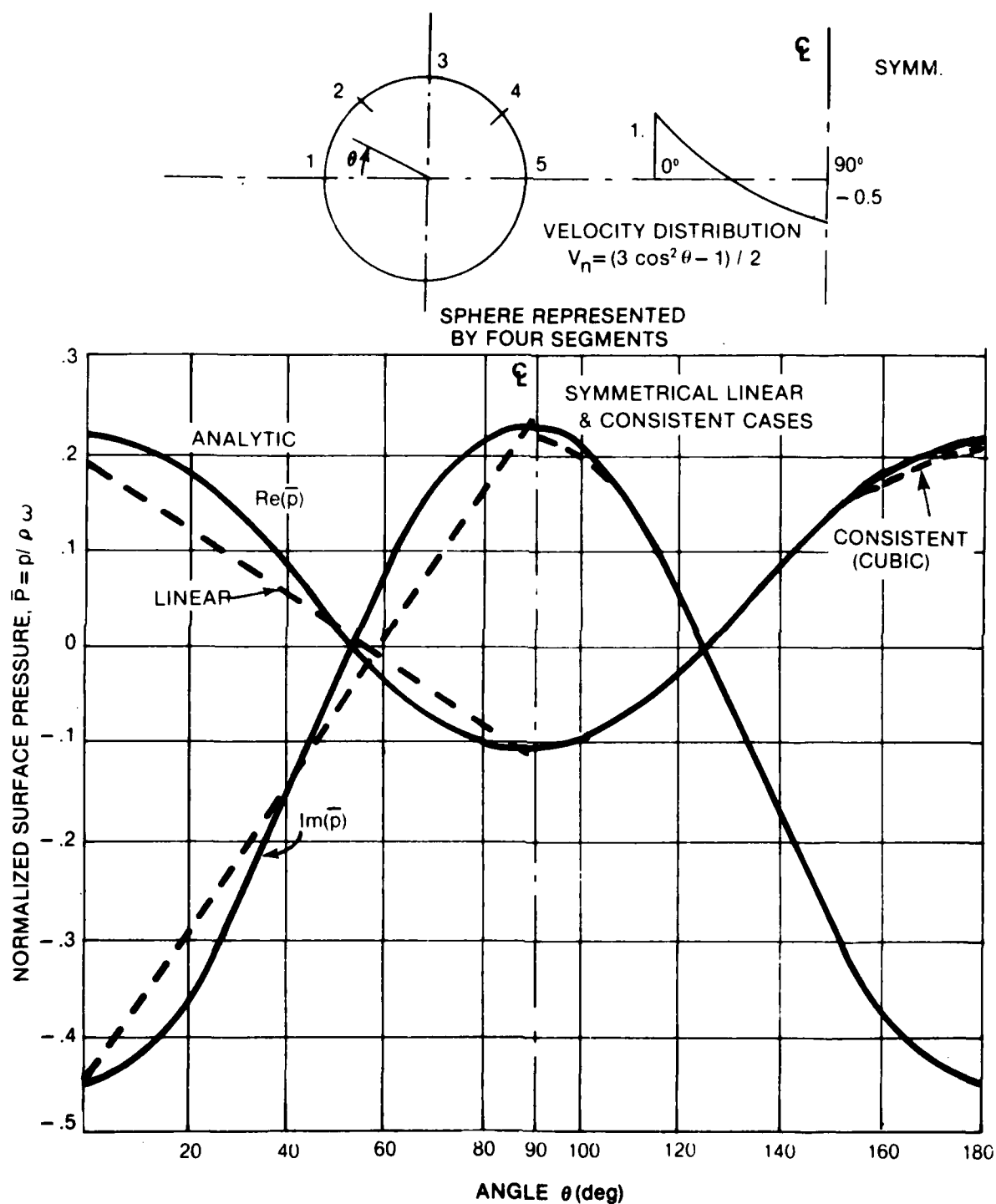


Figure 3. Pressure Distribution on the Surface of the Sphere for a Given Velocity Distribution $ka = 2$

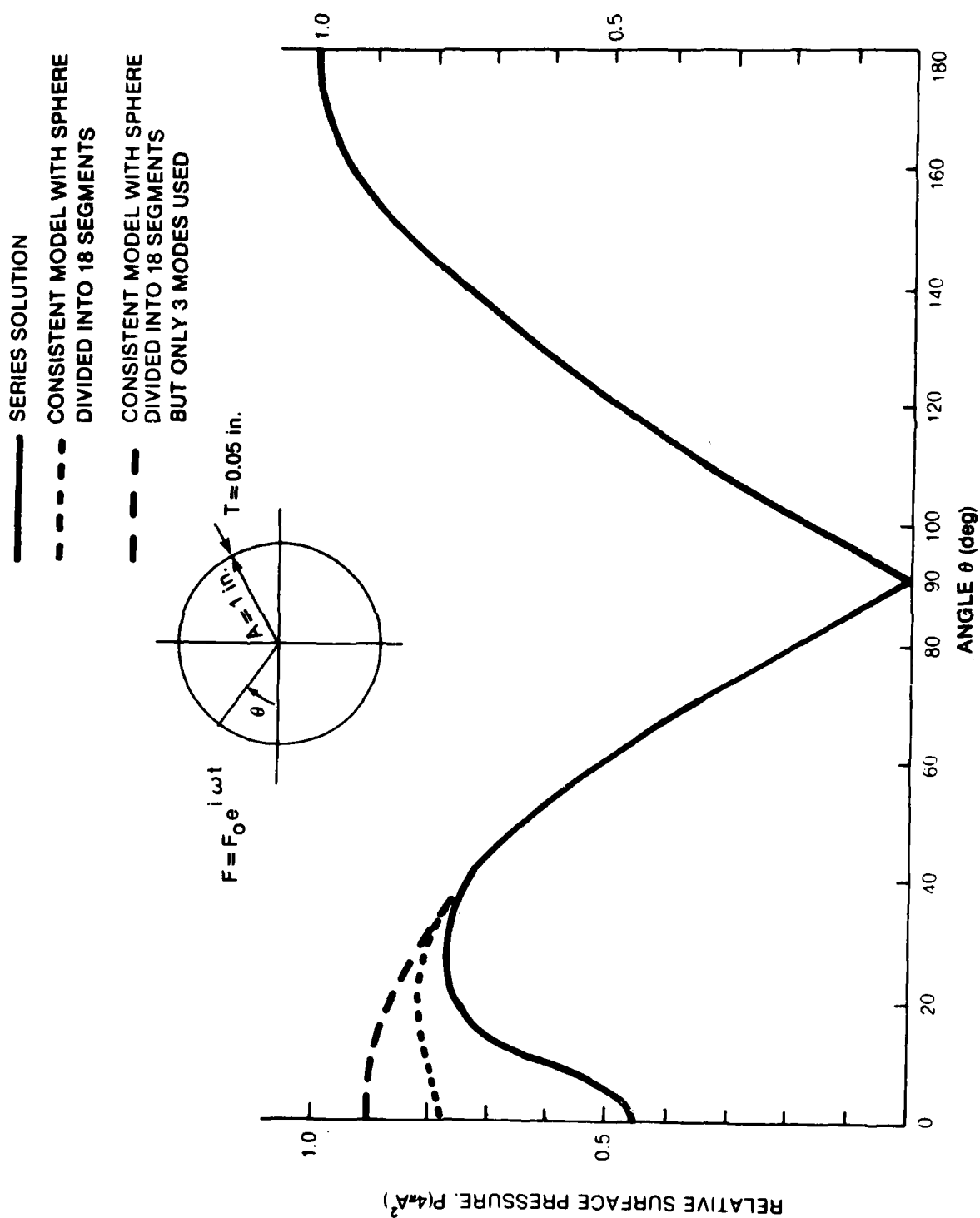


Figure 4. Surface Pressure on an Elastic Spherical Shell Driven by a Point Harmonic Load, $ka = 0.4$

Also, two types of pressure distribution over the shell along the generator are considered: "consistent" and "linear." In the consistent case, the pressure is described by a third order polynomial. In the linear case, the pressure variation is linear and is described by a first order polynomial. In both pressure distribution cases, the velocity variation is given by a third order polynomial.

The geometric shape and driving frequency, as well as the variation of velocity and pressure, determine the coefficients of the [H] and [G] matrices. The size of vector $\{P_n\}$ in the case of consistent pressure distribution is twice that in the linear case. In the consistent case, the variables are the pressure and the spatial derivative of pressure at each grid circle. In case (1), for a given velocity distribution v_n , the analysis gives the pressure distribution for both the consistent and linear cases. For the cases considered, the actual magnitude of pressure in the consistent and linear cases varies little, but since only a few segments are required for the solution, the difference is larger within each segment, as shown in figure 3. Also, the consistent case agrees better with the known exact solution. It is estimated that, for satisfactory results in case (1), the length of each segment should be less than or equal to one-fourth the wavelength of the velocity distribution along the generator of the shell. This observation leads us to believe that the mesh selected for finite element analysis of the structure will prove adequate for the solution of equation (14).

In case (2), which is simply the extension of case (1) since the inclusion is rigid, the net velocity $\{v^n\}$ along the surface is zero. This shows that the scattered velocity $\{v^s\}$ is simply the negative of the incident velocity $\{v^{in}\}$. The fundamental relation (14) gives the pressure caused by a scattered wave. Therefore, The observations for case (1) apply as well to case (2).

Similarly, case (4) is an extension of case (3). In case (3), it is observed that for the solution of equation (35), the available inversion routine does not give a reliable inverted matrix for problems of a size larger than 20. In this case, it is advisable to resort to the solution of equation (39). The results obtained using equation (39) for a spherical shell divided into eight segments and driven at a point compare well with the exact solution everywhere over the sphere except in the vicinity of the point load (figure 4).

Summary

Analysis of scattering and radiation of acoustic waves from an axisymmetric structure in an infinite fluid medium has been presented. It has been shown that the use of a basis function, consistent with the one used for axisymmetric pressure on the finite surface, yields accurate results and uses fewer finite surface elements than does piston-type analysis.

References

1. J. T. Hunt, M. R. Knittel, and D. Barach, "Finite Element Approach to Acoustic Radiation from Elastic Structures," *Journal of the Acoustical Society of America*, vol. 55, no. 2, 1974, pp. 269-280.
2. G. Chertock, "Sound Radiation from Vibrating Surfaces," *Journal of the Acoustical Society of America*, vol. 36, July 1964, pp. 1305-1313.
3. H. A. Schenck, "Improved Integral Formulation for Acoustic Radiation Problems," *Journal of the Acoustical Society of America*, vol. 44, January 1968, pp. 41-58.
4. L. G. Copley, "Integral Equation Method for Radiation from Vibrating Bodies," *Journal of the Acoustical Society of America*, vol. 41, July 1966, pp. 807-815.
5. L. H. Chen and D. G. Schweikert, "Sound Radiation from an Arbitrary Body," *Journal of the Acoustical Society of America*, vol. 35, October 1963, pp. 1626-1632.
6. E. H. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, England, 1964, p. 195.
7. S. Hayek, "Vibration of a Spherical Shell in an Acoustic Medium," *Journal of the Acoustical Society of America*, vol. 40, no. 2, 1966.
8. M. Goldstein, "Linear Least Square Estimation Using the Generalized Inverse," Technical paper of UNIVAC Scientific Exchange Conference, Atlanta, GA, October 12-16, 1970, pp. 355-384.
9. M. Goldstein, Solving System of Linear Equations by Generalized Inverse, Report No. 874, Naval Underwater Systems Center, New London, CT, March 1968.
10. J. L. Hess, Calculation of Acoustic Fields About Arbitrary Three-Dimensional Bodies by a Method of Surface Source Distributions Based on Certain Wave Number Expansions, Report No. DAC 66901, McDonald-Douglas Corp., March 1968.
11. R. R. Smith, J. T. Hunt, and D. Barach, "Finite Element Analysis of Acoustically Radiating Structures with Application to Sonar Transducers," *Journal of the Acoustical Society of America*, vol. 54, May 1973, pp. 1277-1288.
12. J. J. Englem and R. B. Nelson, "Consistent Formulation of Sound Radiation from Arbitrary Structures," *Journal of Applied Mechanics Transactions of the ASME*, June 1975, pp. 295-300.

Appendix A

Analysis of Axisymmetric Shells

Introduction

Normally, an axisymmetric shell is idealized into a series of axisymmetric conical shell finite elements. The finite surface elements for the solution of the Helmholtz surface integral are selected here to be the axisymmetric conical segments. The basis function that defines the velocity distribution on a conical segment is selected to be the same as the basis function used for axisymmetric conical shell finite elements.

In this section, the directional derivatives of $\exp(ikR)/R$, which are used in equations (11) and (12), and their numerical solutions are dealt with, along with the derivation of generalized fluid forces and the consistent mass matrix.

Main Analysis

An arbitrary pressure and velocity distribution on the surface of the cone can be described in a series form:

$$p(s, \phi) = \sum_{m=0}^{\infty} p_m(s) \cos m\phi + \sum_{m=1}^{\infty} p_m^*(s) \sin m\phi \quad (\text{A-1})$$

$$v(s, \phi) = \sum_{m=0}^{\infty} v_m(s) \cos m\phi + \sum_{m=1}^{\infty} v_m^*(s) \sin m\phi : \quad (\text{A-2})$$

We will consider the case that requires the pressure distribution corresponding to a velocity distribution given by only one term of the infinite series, say, $v_m(s) \cos m\phi$:

$$p(s, \phi) = p_m(s) \cos m\phi \quad (\text{A-3})$$

$$v(s, \phi) = v_m(s) \cos m\phi. \quad (\text{A-4})$$

From now on we will drop the subscript m of $p_m(s)$ and $v_m(s)$, and we will evaluate integral equations (5) and (8) for an axisymmetric structure, as shown in figure A-1:

$$p(\underline{x}') = \frac{1}{2\pi} \int_A \left\{ p(\underline{x}) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - ik\rho c v(\underline{x}) \frac{e^{ikR}}{R} \right\} dA \quad (\text{A-5})$$

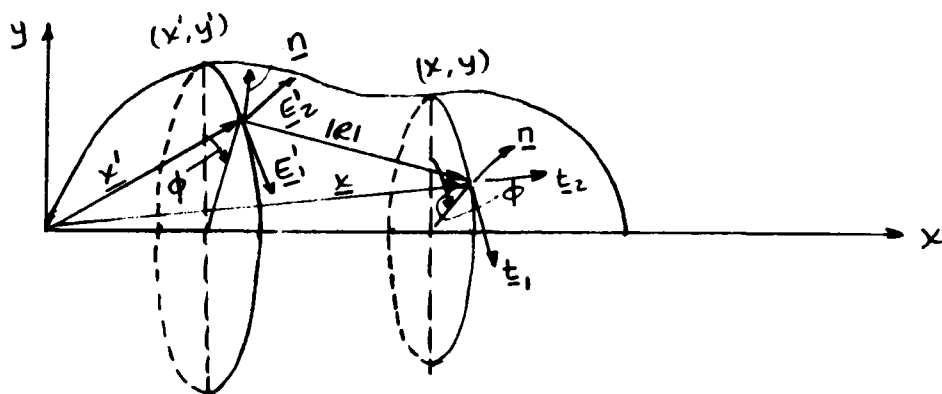


Figure A-1. Schematic for Helmholtz Integrals for an Axisymmetric Structure

$$\frac{\partial}{\partial s_2} (p(\underline{x}')) = \frac{1}{2\pi} \int_A \{ p(\underline{x}) \frac{\partial^2}{\partial n \partial s_2} \left(\frac{e^{ikR}}{R} \right) - ikpcv(\underline{x}) \frac{\partial}{\partial s_2} \left(\frac{e^{ikR}}{R} \right) \} dA$$

$$\begin{aligned} R^2 &= \{ |\underline{x} - \underline{x}'| \cdot |\underline{x} - \underline{x}'| \} = \{ (x - x')^2 + (y \sin \phi - y' \sin \phi')^2 + \\ &\quad (y \cos \phi - y' \cos \phi')^2 \} \\ &= \{ (x - x')^2 + y^2 + y'^2 - 2yy'(\sin \phi \sin \phi' + \cos \phi \cos \phi') \} \\ &= \{ (x - x')^2 + y^2 + y'^2 - 2yy' \cos(\phi - \phi') \}. \end{aligned} \quad (A-6)$$

We will consider that point x' is always along the generator of the shell, which implies that $\phi' = 0$:

$$R^2 = (x - x')^2 + (y - y')^2 + 2yy'(1 - \cos \phi). \quad (A-7)$$

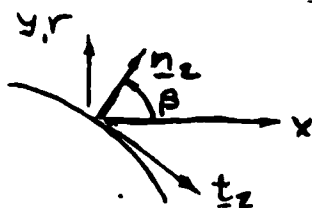
Define the normal derivative:

$$\begin{aligned} \frac{\partial}{\partial n} &= \underline{\nabla} \cdot \underline{n} = \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{\phi r} \frac{\partial}{\partial \phi} + \frac{1}{x} \frac{\partial}{\partial x} \right) \cdot \left(\frac{1}{r} \sin \theta + \frac{1}{x} \cos \theta \right) \\ &= \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial x}. \end{aligned}$$



Define the tangential derivative:

$$\begin{aligned}\frac{\partial}{\partial t_2} &= \nabla \cdot \underline{t}_2 = \left(\underline{i}_r \frac{\partial}{\partial r} + \underline{i}_\phi \frac{1}{r} \frac{\partial}{\partial \phi} + \underline{i}_x \frac{\partial}{\partial x} \right) \cdot \left(-\underline{i}_r \cos \beta + \underline{i}_x \sin \beta \right) \\ &= -\cos \beta \frac{\partial}{\partial r} + \sin \beta \frac{\partial}{\partial x}.\end{aligned}$$



Since the shape is axisymmetric in the above two formulas, we can substitute y for r :

$$\frac{\partial}{\partial n} = \sin \theta \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial x} \quad (\text{A-8})$$

$$\frac{\partial}{\partial t_2} = -\cos \beta \frac{\partial}{\partial y} + \sin \beta \frac{\partial}{\partial x} \quad (\text{A-9})$$

Substituting equations (A-3) and (A-4) into equations (A-5) and (A-6), dropping the subscript m of p , v , and the subscript 2 of t_2 , and letting $\phi' = 0$, we get

$$p(s^1) = \frac{1}{2\pi} \int_0^L \int_{-\pi}^{\pi} \left\{ p(s) \cos m\phi \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - ik\rho cv(s) \cos m\phi \frac{e^{ikR}}{R} \right\} y d\phi ds$$

$$p_{,t}(s^1) = \frac{1}{2\pi} \int_0^L \int_{-\pi}^{\pi} \left\{ p(s) \cos m\phi \frac{\partial^2}{\partial n \partial t_1} \left(\frac{e^{ikR}}{R} \right) - ik\rho cv(s) \cos m\phi \frac{\partial}{\partial t_1} \left(\frac{e^{ikR}}{R} \right) \right\} y d\phi ds$$

Assuming that the total surface is made of J conical finite surface elements, we get

$$p(s_1) = \sum_{j=1}^J \left[\int_0^{L_j} I_1(s_1, s_j) p(s_j) y(s_j) ds_j + \int_0^{L_j} I_2(s_1, s_j) v(s_j) y(s_j) ds_j \right] \quad (\text{A-10})$$

$$p_{,t_1}(s_1) = \sum_{j=1}^J \left[\int_0^{L_j} I_3(s_1, s_j) p(s_j) y(s_j) ds_j + \int_0^{L_j} I_4(s_1, s_j) v(s_j) y(s_j) ds_j \right], \quad (\text{A-11})$$

where

s_i refers to the i th grid circle where pressure is to be found

s_j refers to the field point on the generator of j th conical finite surface element $0 \leq s_j \leq L_j$

$y(s_j)$ is the y coordinate of the field point s_j

$v(s_j)$ is the velocity normal to the surface of the field point s_j .

$$p_{,t_2}(s_i) = \frac{\partial}{\partial t_1} \left\{ p(s_i) \right\} = \frac{\partial}{\partial t_i} \left\{ p(s_i) \right\} \quad (\text{Note the way the index of } t \text{ is changed.})$$

$$I_1(s_i, s_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial n_j} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) \cos m\phi \, d\phi \quad (\text{A-12})$$

$$I_2(s_i, s_j) = - \frac{ik\rho c}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) \cos m\phi \, d\phi \quad (\text{A-13})$$

$$I_3(s_i, s_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial^2}{\partial n_j \partial t_i} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) \cos m\phi \, d\phi \quad (\text{A-14})$$

$$I_4(s_i, s_j) = - \frac{ik\rho c}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial t_i} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) \cos m\phi \, d\phi \quad (\text{A-15})$$

$$\frac{\partial}{\partial n_j} = \sin\theta_j \frac{\partial}{\partial y_j} + \cos\theta_j \frac{\partial}{\partial x_j} \quad (\text{A-16})$$

$$\frac{\partial}{\partial t_i} = - \cos\theta_i \frac{\partial}{\partial y_i} + \sin\theta_i \frac{\partial}{\partial x_i} \quad (\text{A-17})$$

$$R_{ij} = \{ (x_j - x_i)^2 + (y_j - y_i)^2 + 2y_i y_j (1 - \cos\phi) \}^{1/2} \quad (\text{A-18})$$

Polynomial Evaluation

Given the slopes β_j^1 , β_j^2 and the ordinates Y_j and Y_{j+1} , and if the curve is defined by a third order polynomial

$$y_j = c_0 + c_1 y_j + c_2 y_j^2 + c_3 y_j^3,$$

coefficients c_0 through c_3 can be determined by the four boundary conditions β_j^1 , β_j^2 , Y_j , Y_{j+1} . (See figure A-2.) The resulting expression can be rearranged so that

$$y_j = (1-3x^2+2x^3)Y_j + L_j(x-2x^2+x^3)\beta_j + (3x^2-2x^3)Y_{j+1} + L_j(-x^2+x^3)\beta_{j+1}, \quad (\text{A-19})$$

where

$$x = \frac{s_j}{L_j}.$$

Rewritten,

$$y_j = A1_j(1)Y_j + A1_j(2)\beta_j + A2_j(1)Y_{j+1} + A2_j(2)\beta_{j+1}, \quad (\text{A-20})$$

where

$$\begin{aligned} A1_j(1) &= 1-3x^2+2x^3 & A2_j(1) &= 3x^2-2x^3 \\ A1_j(2) &= (x-2x^2+x^3)L_j & A2_j(2) &= (-x^2+x^3)L_j. \end{aligned} \quad (\text{A-21})$$

$$y_j = Y_j + \bar{y}_j \cos \alpha_j^2 + s_j \sin \alpha_j^2, \quad (\text{A-22})$$

where

$$\bar{y}_j = A1_j(2)\beta_j^1 + A2_j(2)\beta_j^2 \quad (\text{A-23})$$

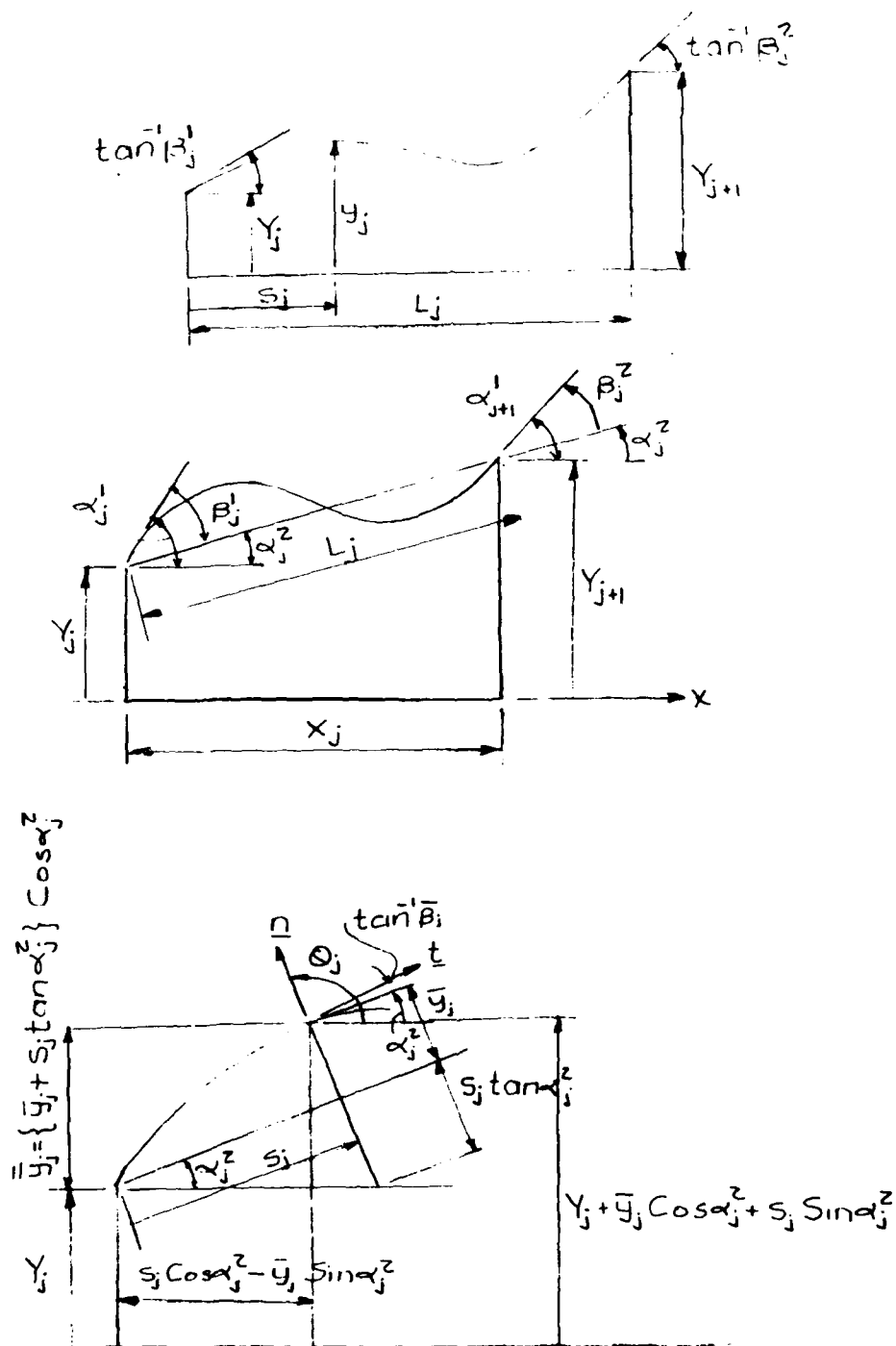


Figure A-2. Schematics for Cubic Polynomial

$$\bar{\beta}_j = \frac{d}{ds_j} \bar{y}_j = (1-4x+3x^2)\beta_j^1 + (-2x+3x^2)\beta_j^2 \quad (\text{A-24})$$

$$\theta_j = 90 + \alpha_j^2 + \tan^{-1} \bar{\beta}_j. \quad (\text{A-25})$$

Velocity Distribution

$$\begin{aligned} v(s_j) &= A1_j(1)\dot{W}_j + A1_j(2)\dot{\alpha}_j + A2_j(1)\dot{W}_{j+1} + A2_j(2)\dot{\alpha}_{j+1} \\ &= \{A1_j\}^T \{D_j\} + \{A2_j\}^T \{D_{j+1}\}, \end{aligned} \quad (\text{A-26})$$

where W_j = displacement normal to the surface at jth grid circle.

$$\begin{aligned} \alpha_j &= \frac{d}{ds_j} W_j & \{A1_j\}^T &= \{A1_j(1), A1_j(2)\} \\ \dot{W}_j &= \frac{d}{dt} W_j & \{A2_j\}^T &= \{A2_j(1), A2_j(2)\} \\ \{D_j\} &= \left\{ \begin{array}{c} \dot{W}_j \\ \dot{\alpha}_j \end{array} \right\}. \end{aligned}$$

Pressure Distribution

$$\begin{aligned} p(s_j) &= A1_j(1)P_j + A1_j(2)P_j^1 + A2_j(1)P_{j+1} + A2_j(2)P_{j+1}^1 \\ &= \{A1_j\}^T \{S_j\} + \{A2_j\}^T \{S_{j+1}\}, \end{aligned} \quad (\text{A-27})$$

where P_j = pressure normal to the surface at jth grid circle.

$$P_j^1 = \frac{d}{ds_j} P_j.$$

If only the linear variation is to be used,

$$p(s_j) = c1_j P_j + c2_j P_{j+1}, \quad (\text{A-28})$$

where

$$c1_j = 1 - \frac{s_j}{L_j} \quad c2_j = \frac{s_j}{L_j}. \quad (\text{A-29})$$

Evaluation of $I_i(s_i, s_j)$ **Evaluation of Directional Derivatives**

$$\begin{aligned}
 \frac{e^{ikR_{ij}}}{R_{ij}} &= \frac{\cos kR_{ij} + \sin kR_{ij}}{R_{ij}} \\
 \frac{\partial}{\partial n_j} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) &= \frac{\partial}{\partial R_{ij}} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) \frac{\partial R_{ij}}{\partial n_j} \\
 \frac{\partial}{\partial t_i} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) &= \frac{\partial}{\partial R_{ij}} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) \frac{\partial R_{ij}}{\partial t_i} \\
 \frac{\partial}{\partial R_{ij}} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) &= \frac{1}{R_{ij}^2} \{ R_{ij} \cdot ike^{ikR_{ij}} - e^{ikR_{ij}} \} \\
 &= ik \frac{e^{ikR_{ij}}}{R_{ij}} - \frac{e^{ikR_{ij}}}{R_{ij}^2} \quad (A-30)
 \end{aligned}$$

From equations (A-16) and (A-17),

$$\begin{aligned}
 \frac{\partial}{\partial n_j} R_{ij} &= \frac{\partial}{\partial x_j} R_{ij} \cdot \cos \theta_j + \sin \theta_j \frac{\partial R_{ij}}{\partial y_j} \\
 \frac{\partial}{\partial t_i} R_{ij} &= -\cos \theta_i \frac{\partial R_{ij}}{\partial y_i} + \sin \theta_i \frac{\partial R_{ij}}{\partial x_j} \\
 R_{ij}^2 &= (x_j - x_i)^2 + (y_j - y_i)^2 + 2y_i y_j (1 - \cos \phi) \\
 2R_{ij} \frac{\partial R_{ij}}{\partial x_j} &= 2(x_j - x_i) \\
 \frac{\partial R_{ij}}{\partial x_j} &= \frac{x_j - x_i}{R_{ij}}
 \end{aligned}$$

$$\frac{\partial R_{ij}}{\partial x_i} = \frac{x_i - x_j}{R_{ij}}$$

$$\frac{\partial R_{ij}}{\partial y_j} = \frac{1}{R_{ij}} \{y_j - y_i \cos \phi\}$$

$$\frac{\partial R_{ij}}{\partial y_i} = \frac{1}{R_{ij}} \{y_i - y_j \cos \phi\}$$

$$\frac{\partial}{\partial n_j} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) = (ikR_{ij} - 1) \frac{e^{ikR_{ij}}}{R_{ij}^2} \left\{ \cos \theta_j \left(\frac{x_j - x_i}{R_{ij}} \right) + \sin \theta_j \left(\frac{y_j - y_i \cos \phi}{R_{ij}} \right) \right\} \quad (A-31)$$

$$\frac{\partial}{\partial t_i} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) = (ikR_{ij} - 1) \frac{e^{ikR_{ij}}}{R_{ij}^2} \left\{ -\cos \theta_i \frac{y_i - y_j \cos \phi}{R_{ij}} + \sin \theta_i \frac{x_i - x_j}{R_{ij}} \right\} \quad (A-32)$$

Simplifying the terms shown below, we get

$$\begin{aligned} (ikR_{ij} - 1) \frac{e^{ikR_{ij}}}{R_{ij}^2} &= \left(\frac{ik}{R_{ij}} - \frac{1}{R_{ij}^2} \right) (\cos kR_{ij} + i \sin kR_{ij}) \\ &= \left(-\frac{1}{R_{ij}^2} \cos kR_{ij} - \frac{k}{R_{ij}} \sin kR_{ij} \right) + i \left(\frac{k}{R_{ij}} \cos kR_{ij} - \frac{1}{R_{ij}^2} \sin kR_{ij} \right) \\ &= - \left(\frac{\cos kR_{ij} + kR_{ij} \sin kR_{ij}}{R_{ij}^2} \right) + i \left(\frac{kR_{ij} \cos kR_{ij} - \sin kR_{ij}}{R_{ij}^2} \right) \quad (A-33) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial R_{ij}} \left\{ \left(\frac{ik}{R_{ij}^2} - \frac{1}{R_{ij}^3} \right) e^{ikR_{ij}} \right\} &= \left\{ ik \left(\frac{ik}{R_{ij}^2} - \frac{1}{R_{ij}^3} \right) + \left(-\frac{2ik}{R_{ij}^3} + \frac{3}{R_{ij}^4} \right) \right\} e^{ikR_{ij}} \\ &= \left\{ -\frac{k^2}{R_{ij}^2} + \frac{3}{R_{ij}^4} - i \left(\frac{3k}{R_{ij}^3} \right) \right\} (\cos kR_{ij} + i \sin kR_{ij}) \\ &= \left\{ \left(\frac{3 - k^2 R_{ij}^2}{R_{ij}^4} \right) \cos kR_{ij} + \frac{3k}{R_{ij}^3} \sin kR_{ij} \right\} + i \left\{ \left(\frac{3 - k^2 R_{ij}^2}{R_{ij}^4} \right) \sin kR_{ij} - \frac{3k}{R_{ij}^3} \cos kR_{ij} \right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial n_j \partial t_i} \left(\frac{e^{ikR_{ij}}}{R_{ij}} \right) &= \left[\left\{ \left(\frac{3 - k^2 R_{ij}^2}{R_{ij}^4} \right) \cos kR_{ij} + \frac{3k}{R_{ij}^3} \sin kR_{ij} \right\} + \right. \\ &\quad \left. i \left\{ \left(\frac{3 - k^2 R_{ij}^2}{R_{ij}^4} \right) \sin kR_{ij} - \frac{3k}{R_{ij}^3} \cos kR_{ij} \right\} \right] \{ -(y_i - y_j \cos \phi) \cos \theta_i + \end{aligned}$$

$$(x_i - x_j) \sin \theta_i \} \cdot \frac{1}{R_{ij}} \{ (x_j - x_i) \cos \theta_j + (y_j - y_i \cos \phi) \sin \theta_j \}. \quad (\text{A-34})$$

Substituting equations (A-30) through (A-34) into equations (A-12) through (A-15) and separating the real and imaginary parts, we can write

$$I_1(s_i, s_j) = I_{11}(s_i, s_j) + iI_{12}(s_i, s_j) \quad (\text{A-35})$$

$$I_2(s_i, s_j) = I_{21}(s_i, s_j) + iI_{22}(s_i, s_j) \quad (\text{A-36})$$

$$I_3(s_i, s_j) = I_{31}(s_i, s_j) + iI_{32}(s_i, s_j) \quad (\text{A-37})$$

$$I_4(s_i, s_j) = I_{41}(s_i, s_j) + iI_{42}(s_i, s_j), \quad (\text{A-38})$$

where

$$I_{11} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ -\frac{\cos kR_{ij} + kR_{ij} \sin kR_{ij}}{R_{ij}^3} \right\} \left[(x_j - x_i) \cos \theta_j + (y_j - y_i \cos \phi) \sin \theta_j \right] \cos m\phi d\phi \quad (\text{A-39})$$

$$I_{12}(s_i, s_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{kR_{ij} \cos kR_{ij} - \sin kR_{ij}}{R_{ij}^3} \right\} \cdot \{ (x_j - x_i) \cos \theta_j + (y_j - y_i \cos \phi) \sin \theta_j \} \cos m\phi d\phi \quad (\text{A-40})$$

$$I_{21} = \frac{k\rho c}{2\pi} \int_{-\pi}^{\pi} \frac{\sin kR_{ij}}{R_{ij}} \cos m\phi d\phi \quad (\text{A-41})$$

$$I_{22} = -\frac{k\rho c}{2\pi} \int_{-\pi}^{\pi} \frac{\cos kR_{ij}}{R_{ij}} \cos m\phi d\phi \quad (\text{A-42})$$

$$I_{31} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{(3 - k^2 R_{ij}^2) \cos kR_{ij} + 3kR_{ij} \sin kR_{ij}}{R_{ij}^5} \right\} F1.F2. \cos m\phi d\phi \quad (\text{A-43})$$

$$I_{32} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{(3 - k^2 R_{ij}^2) \sin kR_{ij} - 3kR_{ij} \cos kR_{ij}}{R_{ij}^5} \right\} F1.F2. \cos m\phi d\phi \quad (\text{A-44})$$

$$F1 = (x_i - x_j) \sin \theta_i - (y_i - y_j \cos \phi) \cos \theta_i$$

$$F2 = (x_j - x_i) \cos \theta_j + (y_j - y_i \cos \phi) \sin \theta_j$$

$$I_{41} = \frac{k\rho c}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{kR_{ij} \cos kR_{ij} - \sin kR_{ij}}{R_{ij}^3} \right\}$$

$$\cdot \left\{ (x_i - x_j) \sin \theta_i - (y_i - y_j \cos \phi) \cos \theta_i \right\} \cos m\phi d\phi \quad (A-45)$$

$$I_{42} = - \frac{k\rho c}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\cos kR_{ij} + kR_{ij} \sin kR_{ij}}{R_{ij}^3} \right\}$$

$$\left\{ (x_i - x_j) \sin \theta_i - (y_i - y_j \cos \phi) \cos \theta_i \right\} \cos m\phi d\phi. \quad (A-46)$$

Assumption 1. Consistent Pressure Distribution

Substituting the integrals I_1 through I_4 and equations (A-26) and (A-27) into equations (A-10) and (A-11), we get

$$p(s_i) = \sum_{j=1}^J \left[\{G1(i, j)\}^T \{D_j\} + \{G2(i, j)\}^T \{D_{j+1}\} \right. \\ \left. + \{H1(i, j)\}^T \{S_j\} + \{H2(i, j)\}^T \{S_{j+1}\} \right] \quad (A-47.1)$$

$$P_t(s_i) = \sum_{j=1}^J \left\{ \{G3(i, j)\}^T \{D_j\} + \{G4(i, j)\}^T \{D_{j+1}\} \right. \\ \left. + \{H3(i, j)\}^T \{S_j\} + \{H4(i, j)\}^T \{S_{j+1}\} \right\}. \quad (A-47.2)$$

The two preceding equations (A-47) can be combined in one matrix equation:

$$\{s_i\} = \sum_{j=1}^J \begin{bmatrix} \{G1(i,j)\}^T & \{G2(i,j)\}^T \\ \{G3(i,j)\}^T & \{G4(i,j)\}^T \end{bmatrix} \begin{pmatrix} \{D_j\} \\ \{D_{j+1}\} \end{pmatrix} + \begin{bmatrix} \{H1(i,j)\}^T & \{H2(i,j)\}^T \\ \{H3(i,j)\}^T & \{H4(i,j)\}^T \end{bmatrix} \begin{pmatrix} \{s_j\} \\ \{s_{j+1}\} \end{pmatrix}, \quad (A-48)$$

where $i = 1, 2, \dots, (J + 1)$

$$i = 1, 2, \dots, (J+1)$$

$$\{G1(i,j)\}^T = \int_0^{L_j} I_2(s_i, s_j) y(s_j) \{A1_j\}^T ds_j \quad (A-49.1)$$

$$\{G2(i,j)\}^T = \int_0^{L_j} I_2(s_i, s_j) y(s_j) \{A2_j\}^T ds_j \quad (A-49.2)$$

$$\{G3(i,j)\}^T = \int_0^{L_j} I_4(s_i, s_j) y(s_j) \{A1_j\}^T ds_j \quad (A-49.3)$$

$$\{G4(i,j)\}^T = \int_0^{L_j} I_4(s_i, s_j) y(s_j) \{A2_j\}^T ds_j \quad (A-49.4)$$

$$\{H1(i,j)\}^T = \int_0^{L_j} I_1(s_i, s_j) y(s_j) \{A1_j\}^T ds_j \quad (A-49.5)$$

$$\{H2(i,j)\}^T = \int_0^{L_j} I_1(s_i, s_j) y(s_j) \{A2_j\}^T ds_j \quad (A-49.6)$$

$$\{H3(i,j)\}^T = \int_0^{L_j} I_3(s_i, s_j) y(s_j) \{A1_j\}^T ds_j \quad (A-49.7)$$

$$\{H4(i,j)\}^T = \int_0^{L_j} I_3(s_i, s_j) y(s_j) \{A2_j\}^T ds_j. \quad (A-49.8)$$

Equations (A-48) represent $2(J + 1)$ equations, which are written in matrix form as

$$[E]\{S\} = [G]\{D\} + [H]\{S\}, \quad (A-50)$$

where

$[E]$ is the unit matrix, $(2J + 2)$ size

$[G]$, $[H]$ are complex square matrices, $(2J + 2)$ size

$\{S\}$, $\{D\}$ are the vectors of generalized pressure and velocity, respectively, of $(2J + 2)$ size.

Assumption 2. Linear Pressure Distribution

Substituting the integrals I_1 and I_2 and equations (A-26) and (A-28) into equations (A-10), we get

$$p(s_i) = \sum_{j=1}^J \left[\{G1(i,j)\}^T \{D_j\} + \{G2(i,j)\}^T \{D_{j+1}\} + \bar{H}1(i,j) P_j + \bar{H}2(i,j) P_{j+1} \right], \quad (A-51)$$

where

$$\bar{H}1(i,j) = \int_0^{L_j} I_1(s_i, s_j) y(s_j) c1_j ds_j \quad (A-52.1)$$

$$\bar{H}2(i,j) = \int_0^{L_j} I_1(s_i, s_j) y(s_j) c2_j ds_j, \quad (A-52.2)$$

where

$$c1_j = 1 - \frac{s_j}{L_j}, \quad c2_j = \frac{s_j}{L_j}.$$

The total of $(J + 1)$ equations (A-51) are written in matrix form as

$$[\bar{E}]\{P\} = [\bar{G}]\{D\} + [\bar{H}]\{P\}, \quad (A-53)$$

where

$[\bar{E}]$ is a unit square matrix of $(J + 1)$ size

$[\bar{G}]$ is the complex matrix of $(J + 1)$, $(2J + 2)$ size

$[\bar{H}]$ is the square complex matrix of $(J + 1)$ size

$\{P\}$ is the vector of pressures, $(J + 1)$ size

$\{D\}$ is the vector of generalized velocity of $(2J + 2)$ size.

Numerical Integration

It is apparent that the integrals (A-40) through (A-47), (A-49•i(i = 1,8)), and (A-52•i(i = 1,2)) can be evaluated only by using some numerical scheme. Here the Gaussian integration formula is used because the integrands are analytical expressions, which makes it easy to calculate their values at any required point, and because the number of calculations to be performed for a required accuracy are fewer. The Gaussian integration method reduces the integral to the summation of the weighted values of the integral at the preassigned number of points. The Gauss-Legendre integration formula is expressed as

$$\int_{-1}^1 y(x) dx = \sum_{i=1}^n A_i y(x_i). \quad (A-54)$$

Points x_i and the corresponding weighting function A_i are given in standard references on numerical analysis. The integrals (A-49) and (A-52) are not in the form of equation (A-54); as a result, the integrals require transformation. The above integrals are in the form

Let

$$\int_0^L f(s) ds.$$

$$s = \frac{L}{2}(1+s^1), \quad (A-55)$$

which transforms the above integral to

$$\int_0^L f(s) ds = \frac{L}{2} \int_{-1}^1 f\left(\frac{L}{2}(1+s^1)\right) ds^1 = \frac{L}{2} \sum_{i=1}^n A_i f\left(\frac{L}{2}(1+s_i^1)\right). \quad (A-56)$$

Since $f(R_{ij}(\phi))$ and $\cos m\phi$ are both symmetric, integrals (A-40) through (A-47) have the following property

$$\int_{-\pi}^{\pi} f(R_{ij}(\phi)) \cos m\phi d\phi = 2 \int_0^{\pi} f(R_{ij}(\phi)) \cos m\phi d\phi.$$

In order to use the Gauss-Legendre integration formula for an alternating integral like the above, we should first reduce the integral to the sum of the integrals over each wavelength and then use the n-point Gauss-Legendre integration formula over each of the subregions:

$$2 \int_0^{\pi} f(R_{ij}) \cos m\phi d\phi = 2 \sum_{i=1}^m \int_{\frac{\pi}{m}(i-1)}^{\frac{\pi}{m}i} f(R_{ij}) \cos m\phi d\phi.$$

Let

$$\psi = \left(\frac{m\phi}{\pi} - i + 1/2\right)2, \quad \phi = \frac{\pi}{2m}(\psi + 2i - 1).$$

Then

$$\int_{\frac{\pi}{m}(i-1)}^{\frac{\pi}{m}i} f(Rij) \cos m\phi d\phi = \frac{\pi}{2m} \int_{-1}^1 f(Rij(\psi)) \cos\left(\frac{\pi}{2}(\psi+2i-1)\right) d\psi,$$

where

$$\int_{-\pi}^{\pi} f(Rij(\phi)) \cos m\phi d\phi = \sum_{i=1}^m \frac{\pi}{m} \int_{-1}^1 f_1(\psi) d\psi = \sum_{i=1}^m \frac{\pi}{m} \sum_{j=1}^n A_i f_1(\psi_j)$$

where

$$f_1(\psi_j) = f(Rij(\psi_j)) \cos\left(\frac{\pi}{2}(\psi_j+2i-1)\right).$$

In this analysis we have used $n = 2$; therefore,

$$\int_{-\pi}^{\pi} f(Rij(\phi)) \cos m\phi d\phi = \sum_{i=1}^m \frac{\pi}{m} (A_1 f_1(\psi_1) + A_2 f_1(\psi_2)), \quad (A-57)$$

for which

$$A_1 = A_2 = 1$$

$$\psi_1 = 0.57735027$$

$$\psi_2 = -0.57735027$$

$$\phi_1 = \frac{\pi}{m} \left(\frac{\psi_1}{2} + i-1/2 \right) = \frac{\pi}{m} (i-0.21132487)$$

$$\phi_2 = \frac{\pi}{m} \left(\frac{\psi_2}{2} + i-1/2 \right) = \frac{\pi}{m} (i-0.788675).$$

Generalized Forces

The acoustic pressure distributed over each finite surface element has to be expressed as the joint forces $\{F_{jn}\}$, as shown in equation (16).

Total virtual work equals $\int_{-\pi}^{\pi} \int_0^L p \cdot \delta u \cdot y d\phi ds$

$$= \sum_{j=0}^{\infty} \{ Q_1^{1m} \delta \dot{w}_j^m + Q_1^{2m} \delta \dot{\alpha}_j^m + Q_2^{1m} \delta \dot{w}_{j+1}^m + Q_2^{2m} \delta \dot{\alpha}_{j+1}^m \} +$$

$$\sum_{j=0}^{\infty} \{ Q_1^{1m*} \delta \dot{w}_j^{m*} + Q_1^{2m*} \delta \dot{\alpha}_j^{m*} + Q_2^{1m*} \delta \dot{w}_{j+1}^{m*} + Q_2^{2m*} \delta \dot{\alpha}_{j+1}^{m*} \}. \quad (A-58)$$

Substituting equations (A-1) and (A-2) into equation (A-58), using the relations of equation (A-26) and

$$\int_{-\pi}^{\pi} \cos m\phi \cos n\phi d\phi = \epsilon \pi \delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin m\phi \sin n\phi d\phi = \epsilon \pi \delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin m\phi \cos n\phi d\phi = 0$$

$$\text{where } \epsilon = \begin{cases} 2 & m = 0 \\ 1 & m \neq 0 \end{cases}$$

and then comparing the coefficients of the like virtual generalized displacements, we get the expressions for the generalized forces, as shown in figure A-3: (Note that superscript m is dropped and j is added.)

$$Q_1^{1j} = \epsilon \pi \int_0^L p_m(s) y(s) A1_j(1) ds$$

$$Q_1^{2j} = \epsilon \pi \int_0^L p_m(s) y(s) A1_j(2) ds$$

$$Q_2^{1j} = \epsilon \pi \int_0^L p_m(s) y(s) A2_j(1) ds$$

$$Q_2^{2j} = \epsilon \pi \int_0^L p_m(s) y(s) A2_j(2) ds. \quad (A-59)$$

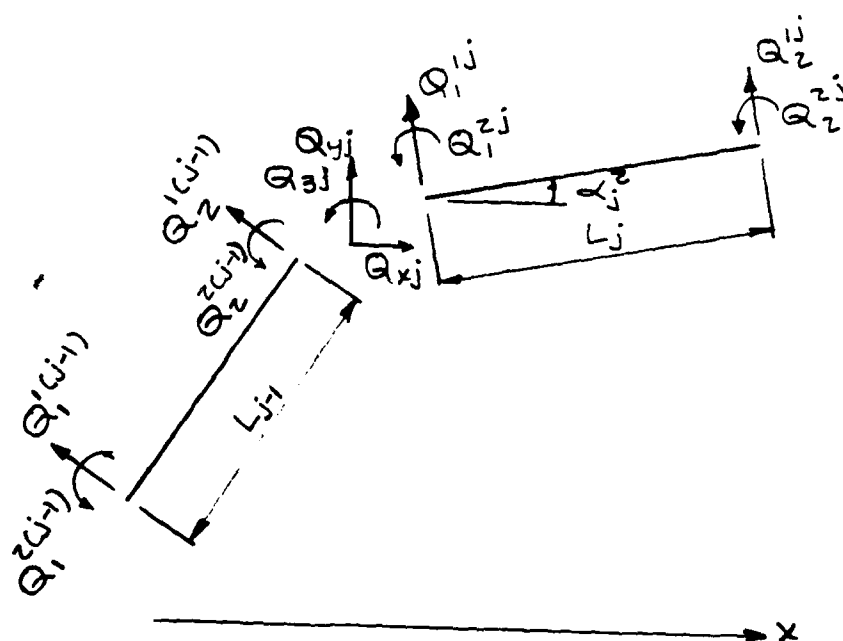
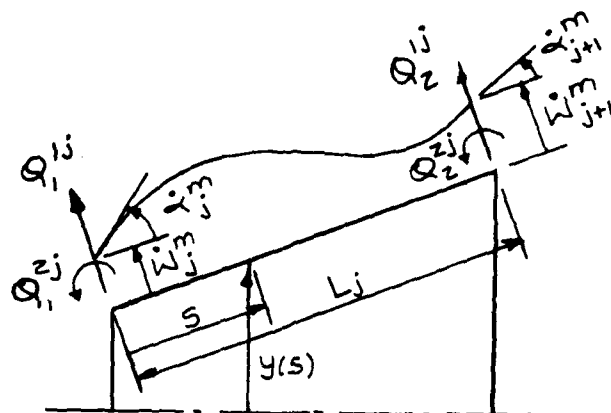


Figure A-3. Generalized Coordinates

In the above equations, the only assumptions we have made concern the velocity distribution, given by equation (A-26), and the definition of the positive direction of the velocity and pressure (along the outer normal to the surface). Although the generator of the axisymmetric segment is shown to be straight, equations (A-59) admit any variation of y , straight or curved. No restriction on the pressure variation of the axisymmetric finite surface is imposed in equations (A-59).

Depending upon whether the pressure distribution is consistent (equation (A-27)) or linear (equation (A-28)), one can analyze the integrals of equations (A-59) exactly if the analytic expression for $y(s)$ is available (e.g., the generator is straight) or numerically if $y(s)$ is arbitrary. In the analysis that follows, the generator of the segment is assumed to be straight, which means

$$y(s) = C1_j \cdot Y_j + C2_j \cdot Y_{j+1}. \quad (A-60)$$

$C1_j$, $C2_j$ are given by equation (A-29).

Assumption 1. Consistent Pressure Distribution

As defined previously,

$$\begin{aligned} p_m(s) &= A1_j(1)P_j + A1_j(2)P_j^1 + A2_j(1)P_{j+1} + A2_j(2)P_{j+1}^1 \\ &= \{A1_j\}^T \{S_j\} + \{A2_j\}^T \{S_{j+1}\}. \end{aligned} \quad (A-27)$$

Substituting equations (A-60) and (A-27) into the integrals of equation (A-59), we get

$$\begin{Bmatrix} Q_1^{1j} \\ Q_1^{2j} \\ Q_2^{1j} \\ Q_2^{2j} \end{Bmatrix} = \epsilon \pi \begin{bmatrix} L_j \left(\frac{2Y_j}{7} + \frac{3Y_{j+1}}{35} \right) & L_j^2 \left(\frac{Y_j}{28} + \frac{Y_{j+1}}{60} \right) & \frac{9L_j}{140} \left(Y_j + Y_{j+1} \right) & -L_j^2 \left(\frac{Y_j}{60} + \frac{Y_{j+1}}{70} \right) \\ L_j^2 \left(\frac{Y_j}{28} + \frac{Y_{j+1}}{60} \right) & L_j^3 \left(\frac{Y_j}{168} + \frac{Y_{j+1}}{280} \right) & L_j^2 \left(\frac{Y_j + Y_{j+1}}{70} + \frac{Y_{j+1}}{60} \right) & -L_j^3 \left(\frac{Y_j}{280} + \frac{Y_{j+1}}{168} \right) \\ \frac{9L_j}{140} \left(Y_j + Y_{j+1} \right) & L_j^2 \left(\frac{Y_j}{70} + \frac{Y_{j+1}}{60} \right) & L_j \left(\frac{3Y_j}{35} + \frac{2Y_{j+1}}{7} \right) & -L_j^2 \left(\frac{Y_j}{60} + \frac{Y_{j+1}}{28} \right) \\ -L_j^2 \left(\frac{Y_j}{60} + \frac{Y_{j+1}}{70} \right) & -L_j^3 \left(\frac{Y_j}{280} + \frac{Y_{j+1}}{168} \right) & -L_j^2 \left(\frac{Y_j + Y_{j+1}}{60} + \frac{Y_{j+1}}{28} \right) & L_j^3 \left(\frac{Y_j}{280} + \frac{Y_{j+1}}{168} \right) \end{bmatrix} \begin{Bmatrix} P_j \\ P_j^1 \\ P_{j+1} \\ P_{j+1}^1 \end{Bmatrix}. \quad (A-61)$$

As explained in equation (17), the above matrix for the j th element can be transformed into the basic coordinate system, so that the total matrix can be built from these element matrices. Here, however, an alternative approach is taken to write explicitly the total matrix. At the j th node point, we can write the following relation:

$$\begin{aligned}
 Q_{xj} &= -\left\{ Q_1^{1j} \sin^2 \alpha_j + Q_2^{1(j-1)} \sin^2 \alpha_{j-1} \right\} \\
 Q_{yj} &= \left\{ Q_2^{1j} \cos^2 \alpha_j + Q_2^{1(j-1)} \cos^2 \alpha_{j-1} \right\} \\
 Q_{3j} &= Q_1^{2j} + Q_2^{2(j-1)}.
 \end{aligned}
 \tag{A-62}$$

Substituting equation (A-61) into (A-62) and rearranging the terms, we can write

$$\begin{aligned}
 Q_{yj} &= A(3j-2, 2j-3)P_{j-1} + A(3j-2, 2j-2)P_{j-1}^1 + A(3j-2, 2j-1)P_j \\
 &\quad + A(3j-2, 2j)P_j^1 + A(3j-2, 2j+1)P_{j+1} + A(3j-2, 2j+2)P_{j+1}^1 \\
 Q_{xj} &= A(3j-1, 2j-3)P_{j-1} + A(3j-1, 2j-2)P_{j-1}^1 + A(3j-1, 2j-1)P_j \\
 &\quad + A(3j-1, 2j)P_j^1 + A(3j-1, 2j+1)P_{j+1} + A(3j-1, 2j+2)P_{j+1}^1
 \end{aligned}
 \tag{A-63}$$

$$\begin{aligned}
 Q_{3j} &= A(3j, 2j-3)P_{j-1} + A(3j, 2j-2)P_{j-1}^1 + A(3j, 2j-1)P_j \\
 &\quad + A(3j, 2j)P_j^1 + A(3j, 2j+1)P_{j+1} + A(3j, 2j+2)P_{j+1}^1
 \end{aligned}$$

$$A(3j-2, 2j-3) = \frac{9}{140} Z_{j-1} (Y_{j-1} + Y_j)$$

$$A(3j-2, 2j-2) = L_{j-1} Z_{j-1} \left(\frac{Y_{j-1}}{70} + \frac{Y_j}{60} \right)$$

$$A(3j-2, 2j-1) = \frac{3}{35} Z_{j-1} Y_{j-1} + \frac{2}{7} Y_j (Z_{j-1} + Z_j) + \frac{3}{35} Z_j Y_{j+1}$$

$$A(3j-2, 2j) = -L_{j-1} Z_{j-1} \left(\frac{Y_{j-1}}{60} + \frac{Y_j}{28} \right) + L_j Z_j \left(\frac{Y_j}{28} + \frac{Y_{j+1}}{60} \right)$$

$$A(3j-2, 2j+1) = \frac{9}{140} Z_j (Y_j + Y_{j+1})$$

$$A(3j-2, 2j+2) = -L_j Z_j \left(\frac{Y_j}{60} + \frac{Y_{j+1}}{70} \right)$$

$$A(3j-1, 2j-3) = \frac{9}{140} R_{j-1} \left(Y_{j-1} + Y_j \right)$$

$$A(3j-1, 2j-2) = L_{j-1} R_{j-1} \left(\frac{Y_{j-1}}{70} + \frac{Y_j}{60} \right)$$

$$A(3j-1, 2j-1) = \frac{3}{35} R_{j-1} Y_{j-1} + \frac{2}{7} Y_j \left(R_{j-1} + R_j \right) + \frac{3}{35} R_j Y_{j+1}$$

$$A(3j-1, 2j) = -L_{j-1} R_{j-1} \left(\frac{Y_{j-1}}{60} + \frac{Y_j}{28} \right) + L_j R_j \left(\frac{Y_j}{28} + \frac{Y_{j+1}}{60} \right)$$

$$A(3j-1, 2j+1) = \frac{9}{140} R_j \left(Y_j + Y_{j+1} \right)$$

$$A(3j-1, 2j+2) = -L_j R_j \left(\frac{Y_j}{60} + \frac{Y_{j+1}}{70} \right)$$

$$A(3j, 2j-3) = -L_{j-1}^2 \left(\frac{Y_{j-1}}{60} + \frac{Y_j}{70} \right) \epsilon \pi$$

$$A(3j, 2j-2) = -\frac{L_{j-1}^3}{280} \left(Y_{j-1} + Y_j \right) \epsilon \pi$$

$$A(3j, 2j-1) = \left\{ -L_{j-1}^2 \left(\frac{Y_{j-1}}{60} + \frac{Y_j}{28} \right) + L_j^2 \left(\frac{Y_j}{28} + \frac{Y_{j+1}}{60} \right) \right\} \epsilon \pi$$

$$A(3j, 2j) = \left\{ L_{j-1}^3 \left(\frac{Y_{j-1}}{280} + \frac{Y_j}{168} \right) + L_j^3 \left(\frac{Y_j}{168} + \frac{Y_{j+1}}{280} \right) \right\} \epsilon \pi$$

$$A(3j, 2j+1) = L_j^2 \left(\frac{Y_j}{70} + \frac{Y_{j+1}}{60} \right) \epsilon \pi$$

$$A(3j, 2j+2) = -\frac{L_j^3}{280} \left(Y_j + Y_{j+1} \right) \epsilon \pi$$

$$\text{where } R_j = \epsilon \pi L_j \sin^2 \alpha_j$$

$$Z_j = \epsilon \pi L_j \cos^2 \alpha_j$$

(A-64)

Substituting equation (A-66) into equation (A-62) and rearranging the terms, we can write

$$\begin{aligned}
 Q_{yj} &= A(3j-2, j-1) P_{j-1} + A(3j-2, j) P_j + A(3j-2, j+1) P_{j+1} \\
 Q_{xj} &= A(3j-1, j-1) P_{j-1} + A(3j-1, j) P_j + A(3j-1, j+1) P_{j+1} \\
 Q_{3j} &= A(3j, j-1) P_{j-1} + A(3j, j) P_j + A(3j, j+1) P_{j+1}
 \end{aligned} \tag{A-67}$$

$$\begin{aligned}
 A(3j-2, j-1) &= Z_{j-1} \left(\frac{Y_{j-1}}{15} + \frac{Y_j}{12} \right) \\
 A(3j-2, j) &= Z_{j-1} \left(\frac{Y_{j-1}}{12} + \frac{4}{15} Y_j \right) + Z_j \left(\frac{4}{15} Y_j + \frac{Y_{j+1}}{12} \right) \\
 A(3j-2, j+1) &= Z_j \left(\frac{Y_j}{12} + \frac{Y_{j+1}}{15} \right) \\
 A(3j-1, j-1) &= R_{j-1} \left(\frac{Y_{j-1}}{15} + \frac{Y_j}{12} \right) \\
 A(3j-1, j) &= R_{j-1} \left(\frac{Y_{j-1}}{12} + \frac{4}{15} Y_j \right) + R_j \left(\frac{4}{15} Y_j + \frac{Y_{j+1}}{12} \right) \\
 A(3j-1, j+1) &= R_j \left(\frac{Y_j}{12} + \frac{Y_{j+1}}{15} \right) \\
 A(3j, j-1) &= -\epsilon \pi L_{j-1}^2 \left(\frac{Y_{j-1}}{60} + \frac{Y_j}{60} \right) \\
 A(3j, j) &= -\epsilon \pi L_{j-1}^2 \left(\frac{Y_{j-1}}{60} + \frac{Y_j}{30} \right) + \epsilon \pi L_j^2 \left(\frac{Y_j}{30} + \frac{Y_{j+1}}{60} \right) \\
 A(3j, j+1) &= \epsilon \pi L_j^2 \left(\frac{Y_j}{60} + \frac{Y_{j+1}}{60} \right)
 \end{aligned}$$

$$\begin{aligned}
 R_j &= -\epsilon \pi L_j \sin^2 \alpha_j \\
 Z_j &= \epsilon \pi L_j \cos^2 \alpha_j
 \end{aligned} \tag{A-68}$$

In the matrix-vector notations, the generalized forces given by equation (A-67) are written as

$$\{F_f\} = [A] \{P\}, \quad (A-69)$$

where

$$\{F_f\} = \begin{Bmatrix} Q_{x1} \\ Q_{y1} \\ Q_{31} \\ \vdots \\ Q_{3J} \end{Bmatrix} \quad \{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \end{Bmatrix}.$$

Matrix $[A]_{3J \times J}$ is highly banded and its coefficients are given by equation (A-68).

Appendix B

Analysis of an Incoming Plane Wave
at an Angle to the Structure

When a plane wave impinges on a surface, the component of the incident plane wave normal to the surface scatters off the surface. Simultaneously, because of this acoustic pressure, the structure, if flexible, vibrates and radiates acoustic waves. (If the structure is rigid, this vibrating radiation component is absent.) The net acoustic pressure is the sum of the incident, scattered, and radiated components. Often the scattered and radiated components are lumped together in the total scattered wave.

Distribution of the incident velocity of a plane wave normal to the surface along the circumference of a circle is expressed in a Fourier series. We obtain the total scattered pressure due to this incident velocity distribution and then add all the components (incident, scattered, and radiated) to get the total acoustic pressure.

The particle velocity of a plane wave traveling along the negative y_1 direction is given by

$$\underline{v}^i = \underline{i}_{y_1} v_o e^{j(ky_1 - \omega t)}, \quad (\text{B-1})$$

where \underline{i}_{y_1} is the unit vector along the y_1 - axis. We will drop $e^{-j\omega t}$ from the analysis that follows.

We can write the transformation of y_1 into x-y coordinates as

$$y_1 = (y - y_o) \cos \alpha - (x - x_o) \sin \alpha. \quad (\text{B-2})$$

As shown in figure B-1,

$$y = r \cos \phi. \quad (\text{B-3})$$

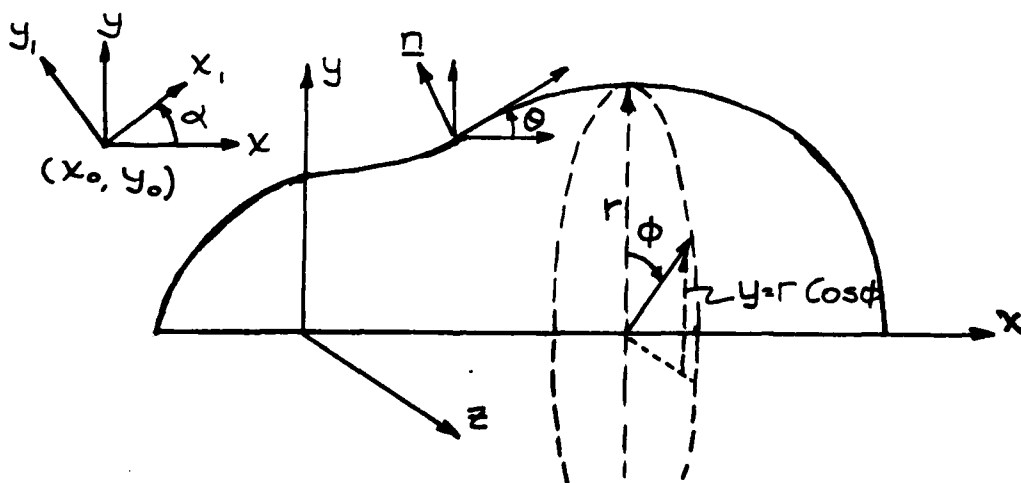


Figure B-1. Definition of Incident Plane Wave With Respect to Axisymmetric Structures

Substituting equation (B-3) into equation (B-2), we can write

$$e^{jky_1} = e^{jk(x_0 \sin \alpha - y_0 \cos \alpha)} \cdot e^{jk(r \cos \alpha \cos \phi)} \cdot e^{-jkx \sin \alpha}. \quad (B-4)$$

For a given x_0, y_0, α , let

$$C = e^{jk(x_0 \sin \alpha - y_0 \cos \alpha)}. \quad (B-5)$$

Also, we can expand¹ the second terms of equation (B-4) into a Fourier Bessel series:

$$e^{jkr \cos \alpha \cos \phi} = \sum_{m=0}^{\infty} \epsilon_m j^m J_m(kr \cos \alpha) \cos m\phi, \quad (B-6)$$

where

$$\epsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \neq 0. \end{cases}$$

Substituting equations (B-5) and (B-6) into equation (B-4) and then into equation (B-1), we get

$$\underline{V}^i = \underline{i}_{y_1} V_0 \cdot C \cdot \left\{ e^{-jkx \sin \alpha} \sum_{m=0}^{\infty} \epsilon_m j^m J_m(kr \cos \alpha) \cos m\phi \right\}. \quad (B-7)$$

We will derive the expression for V^{in} , which is the component of incident velocity normal to the surface:

$$V^{in} = \underline{V}^i \cdot \underline{n} = A \underline{i}_{y_1} \cdot \underline{n},$$

where A is given by equation (B-7) to be

$$A = V_0 C \left\{ e^{-jkx \sin \alpha} \sum_{m=0}^{\infty} \epsilon_m j^m J_m(kr \cos \alpha) \cos m\phi \right\}. \quad (B-8)$$

We can express the unit vector \underline{i}_{y_1} and the unit normal to the surface \underline{n} in x-y basis as

$$\underline{i}_{y_1} = \cos \alpha \underline{i}_y - \sin \alpha \underline{i}_x \quad (B-9)$$

$$\underline{n} = -\sin \theta \underline{i}_x + \cos \theta \cos \phi \underline{i}_y + \cos \theta \sin \phi \underline{i}_z, \quad (B-10)$$

which gives

$$\underline{i}_{y_1} \cdot \underline{n} = \sin \alpha \sin \theta + \cos \theta \cos \alpha \cos \phi. \quad (B-11)$$

Substituting equation (B-11) into the above expression of V^{in} , we get

$$V^{in} = A \{ \sin \alpha \sin \theta + \cos \alpha \cos \theta \cos \phi \}, \quad (B-12)$$

where A is given by equation (B-8).

In the following, we will try to simplify the expression (B-12):

$$\begin{aligned} \frac{v^{in}}{V_0 C} &= \left\{ e^{-jkx \sin \alpha} \sum_{m=0}^{\infty} \epsilon_m j^m J_m(kr \cos \alpha) \cos m\phi \right\} \\ &\quad \{ \sin \alpha \sin \theta + \cos \alpha \cos \theta \cos \phi \} \\ &= \sin \alpha \sin \theta e^{-jkx \sin \alpha} \sum_{m=0}^{\infty} \epsilon_m j^m J_m(kr \cos \alpha) \cos m\phi \\ &\quad + \cos \alpha \cos \theta e^{-jkx \sin \alpha} \sum_{m=0}^{\infty} \epsilon_m j^m J_m(kr \cos \alpha) \frac{1}{2} \{ \cos(m+1)\phi + \cos(m-1)\phi \}. \end{aligned}$$

Rearranging the terms, we can rewrite the above as

$$v^{in} = \sum_{m=0}^{\infty} \epsilon_m j^m \{ A_m(r, x, \theta) \cos(m-1)\phi + B_m(r, x, \theta) \cos m\phi + A_m(r, x, \theta) \cos(m+1)\phi \} \quad (B-13)$$

$$A_m(r, x, \theta) = \frac{V_0 C}{2} \cos \theta \cos \alpha e^{-jkx \sin \alpha} J_m(kr \cos \alpha)$$

$$B_m(r, x, \theta) = V_0 C \sin \theta \sin \alpha e^{-jkx \sin \alpha} J_m(kr \cos \alpha). \quad (B-14)$$

We rewrite equation (B-13) as

$$v^{in} = (B_0 + 2jA_1) + \sum_{m=1}^{\infty} 2j^{m-1} \left\{ A_{m-1} - A_{m+1} + jB_m \right\} \cos m\phi. \quad (B-15)$$

Next we derive the expression for $d/(ds)V^{in}$ by differentiating equation (B-13):

$$\frac{d}{ds} v^{in} = \sum_{m=0}^{\infty} \epsilon_m j^m \left\{ \frac{d}{ds} (A_m) (\cos(m-1)\phi + \cos(m+1)\phi) + \frac{d}{ds} (B_m) \cos m\phi \right\}. \quad (B-16)$$

By the chain rule,

$$\frac{d}{ds} = \frac{\partial x}{\partial s} \cdot \frac{\partial}{\partial x} + \frac{\partial r}{\partial s} \cdot \frac{\partial}{\partial r},$$

but

$$\frac{\partial x}{\partial s} = \cos \theta \quad \text{and} \quad \frac{\partial r}{\partial s} = \sin \theta$$

$$\frac{d}{ds} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial r}.$$

We will use the following relations:

$$\begin{aligned} \frac{\partial}{\partial x} \left\{ e^{-jkx \sin \alpha} J_m(kr \cos \alpha) \right\} &= k \cos \alpha e^{-jkx \sin \alpha} J_m(kr \cos \alpha) \\ \frac{\partial}{\partial r} \left\{ e^{-jkx \sin \alpha} J_m(kr \cos \alpha) \right\} &= k \cos \alpha e^{-jkx \sin \alpha} J_m^1(kr \cos \alpha), \end{aligned}$$

where

$$J_m^1 = \frac{1}{2} \left\{ J_{m-1}(kr \cos \alpha) - J_{m+1}(kr \cos \alpha) \right\} \quad (B-17)$$

to get

$$A_m^1 = \frac{d}{ds} A_m = \frac{V_o C}{2} k \cos \theta \cos \alpha e^{-jkx \sin \alpha} \left\{ -j \sin \alpha \cos \theta J_m + \cos \alpha \sin \theta J_m^1 \right\} \quad (B-18a)$$

$$B_m^1 = \frac{d}{ds} B_m = V_o C k \sin \theta \sin \alpha e^{-jkx \sin \alpha} \left\{ -j \sin \alpha \cos \theta J_m + \cos \alpha \sin \theta J_m^1 \right\}. \quad (B-18b)$$

We rewrite equation (B-16) as

$$\frac{d}{ds} v^{in} = \left\{ B_o^1 + 2jA_1^1 \right\} + \sum_{m=1}^{\infty} 2j^{m-1} \left\{ A_{m-1}^1 - A_{m+1}^1 + jB_m^1 \right\} \cos m\phi. \quad (B-19)$$

Case I. Wave Traveling Along Axis of Symmetry

The expression of the velocity v^{in} and the rate of change of velocity v^{in} can be obtained from equations (B-15) and (B-19) by substituting $\alpha = -90^\circ$, which will align the y_1 - axis with the axis of symmetry:

$$A_m = 0$$

$$B_m = 0 \text{ if } m = 0$$

$$-V_o e^{jk(x-x_o)} \sin \theta \quad \text{if } m = 0.$$

Similarly

$$A_m^1 = 0$$

$$B_m^1 = \begin{cases} 0 & \text{if } m \neq 0 \\ -jkV_o e^{jk(x-x_o)} \sin \theta & \text{if } m = 0. \end{cases}$$

In the above expressions, properties of the Bessel function

$$J_m(0) = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$$

are used.

Expressions (B-15) and (B-19) reduce to

$$v_{in} = B_o = -V_o \sin \theta e^{jk(x-x_o)}$$

$$\frac{d}{ds} v_{in} = v_{in}^1 = B_o^1 = -jkV_o \sin \theta e^{jk(x-x_o)}. \quad (B-20)$$

Case II. Wave Traveling Normal to Axis of Symmetry

If $\alpha = 0$, the y_1 axis coincides with the y axis, and we can get the velocity distribution for a wave traveling along the $-y$ axis. If we want to consider the wave traveling along the $+y$ axis, we have to use $\alpha = 180^\circ$.

In the following we will consider the $\alpha = 180^\circ$ case.

$$\begin{aligned}
 A_m &= -\frac{V_0}{2} \cos\theta (-1)^m e^{jky_0} J_m(kr) \\
 B_m &= 0 \\
 A_m^1 &= \frac{V_0}{2} k \sin\theta \cos\theta e^{jky_0} J_m^1(-kr) \\
 B_m^1 &= 0 \\
 v_{in} &= 2jA_1 + \sum_{m=1}^{\infty} 2j^{m-1} \left(A_{m-1} - A_{m+1} \right) \cos m\phi \\
 v_{in}^1 &= 2jA_1^1 + \sum_{m=1}^{\infty} 2j^{m-1} \left(A_{m-1}^1 - A_{m+1}^1 \right) \cos m\phi.
 \end{aligned} \tag{B-21}$$

Appendix C

Analysis of Scattering of the Plane Wave
From Rigid and Flexible Structures

The net component of the particle velocity normal to the surface \underline{v}^n is equal to the sum of the normal component of the incident velocity and the scattered velocity.

$$\underline{v}^n = \left(\underline{v}^i \cdot \underline{n} \right) \underline{n} + \underline{v}^s, \quad (C-1)$$

where

$$\begin{aligned} \underline{v}^n &= v^n \underline{n}, \quad \underline{v}^i \cdot \underline{n} = v^{in} \\ \underline{v}^s &= v^s \underline{n} \\ v^n &= v^{in} + v^s. \end{aligned} \quad (C-2)$$

Scattered components of the pressure and the velocity, p^s and \underline{v}^s , respectively, satisfy the linear algebraic counterpart of the appropriate boundary value problem² of the Helmholtz equation:

$$[H] \{p^s\} = [G] \{v^s\}. \quad (C-3)$$

Incident velocity and pressure, v^i and p^i , respectively, are related as follows:

$$\begin{array}{c} p^i \rightarrow \boxed{} \rightarrow p^i + p^i_{,x} dx \\ x \rightarrow \boxed{} \rightarrow dx \end{array} \quad \rho_f \frac{\partial v^i}{\partial t} = - \frac{\partial p^i}{\partial x}. \quad (C-4)$$

For harmonic velocity, equation (C-4) reduces to

$$p^i(x, t) = -c \rho_f v^i(x, t). \quad (C-5)$$

The total acoustic pressure at a point is the sum of the total scattered pressures and incident pressure given by equations (C-3) and (C-5), respectively.

Substituting equation (B-23) into equation (B-24), we get

$$\begin{aligned} \{p\} &= \{p^i\} + [H]^{-1} [G] \left\{ \{v^n\} - \{v^{in}\} \right\} \\ p &= -c \rho_f \{v^i\} + [H]^{-1} [G] \{v^n\} - [H]^{-1} [G] \{v^{in}\}. \end{aligned} \quad (C-6)$$

Note that in the above step we constructed the matrix vectors from the set of scalar equations (C-2) and (C-5).

Case I. Rigid Structure

If the structure is assumed rigid, the normal velocity $\{V^n\}$ of the structure is zero, and equation (C-6) reduces to

$$\{p\} = c\rho_f \{V^i\} - [H]^{-1} [G] \{V^{in}\}. \quad (C-7)$$

The listing of the computer program uses this set of equations.

Case II. Flexible Structure

The equations of motion of the finite element model of the structure are written as follows:

$$[M] \{\ddot{X}\} + [K] \{X\} = \{F\} + [A] \{p\}. \quad (C-8)$$

Substituting equation (C-6) into equation (C-8) and noting that the vector $\{V^n\}$ is related to the vector $\{\dot{X}\}$, we can rewrite equation (C-9) as

$$[M] \{\dot{X}\} - [A][H]^{-1}[G]\{\dot{X}\} + [K] \{X\} = \{F\} - c\rho_f [A] \{V^i\} - c\rho_f [A][H]^{-1}[G] \{V^{in}\} \quad (C-9)$$

without introducing the transformation matrix relating the vectors $\{\dot{X}\}$ and $\{V^n\}$.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
CNM, MAT-08T1 (Dr. J. Probus)	1
NRL (Dr. Neubauer)	1
USOC, Code 240	1
NAVELECSYSCOM, PME-124 (CAPT Cox), ELEX 320 (Joel Sinsky) (2)	3
NAVSEASYSCOM, SEA-05H(CAPT Jurgen)	1
DTNSRDC (Dr. Wayne Reader)	1
DTIC	12
DARPA (Dr. Theo Kooij)	2
SSPO (Dr. Phil Selwyn)	1

DATE
FILMED
-8

